Integrals involving triplets of Jacobi and Gegenbauer polynomials and some $3j$-symbols of SO(n), SU(n) and Sp(4)

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The coupling coefficients ($3j$-symbols) for the symmetric (most degenerate) irreducible representations of the orthogonal groups SO(n) in a canonical basis [with SO(n) restricted to SO($n-1$)] and different semicanonical (tree) bases [with SO(n) restricted to SO(n')×SO(n''), $n'+n''=n$] are expressed in terms of the integrals involving triplets of the Gegenbauer and the Jacobi polynomials. The derived usual triple-hypergeometric series (which do not reveal the apparent triangle conditions of the $3j$-symbols) are rearranged directly [without using their relation with the semistretched isofactors of the second kind for the complementary chain Sp(4)⊃SU(2)×SU(2)] into formulas with more rich limits for summation intervals and obvious triangle conditions. The isofactors for the class-one representations of the orthogonal groups and for the class-two representations of the unitary groups (and, of course, the related integrals) turn into the double sums in the cases of the canonical SO(n)⊃SO($n-1$) or U(n)⊃U($n-1$) and semicanonical SO(n)⊃SO($n-2$)×SO(2) chains, as well as into the ${}_4F_3(1)$ series under more specific conditions. Expressions for the most general isofactors of SO(n) for coupling of the two symmetric irreps in the canonical basis are also derived.

1 Introduction

The Clebsch–Gordan (coupling) coefficients and $3j$-symbols (or the Wigner coefficients) of the orthogonal groups SO(n), together with their isoscalar factors (isofactors), maintain great importance in many fields of theoretical physics such as atomic, nuclear and statistical physics. The representation functions in terms of the Gegenbauer (ultraspherical) polynomials are well known for the symmetric (also called most degenerate or class-one) irreducible representations (irreps) of SO(n) in the spherical coordinates (Vilenkin [1]) on the unit sphere S_{n-1}. In particular, the explicit Clebsch–Gordan (CG) coefficients and isofactors of SO(n) in the canonical basis for all three symmetric irreps were considered by Gavrilik [2], Kildyushov and Kuznetsov [3] (see also [4]) and Junker [5], using the direct [2, 5] or rather complicated indirect [3, 4] integration procedures.

Norvaišas and Ališauskas [6] also derived triple-sum expressions for related isofactors of SO(n) in the case of the canonical (labelled by the chain of groups SO(n)⊃SO($n-1$)) and semicanonical bases (labelled by irreps l, l', l'' of the group chains SO(n)⊃SO(n')×SO(n''), $n'+n''=n$, in the polyspherical, or the tree type, coordinates [1, 4, 7]), exploiting the transition matrices [8] (also cf. [9]) between the bases, labelled by the unitary and orthogonal subgroups in the symmetrical irreducible spaces of the U(n) group. They have observed [6, 10] that isofactors for the group chain SO(n)⊃SO(n')×SO(n'') for the coupling of the states of symmetric irreps l_1, l_2 into the states of more general irreps $[L_1L_2]$ are the analytical continuation of the isofactors for the chain Sp(4)⊃SU(2)×SU(2),

$$
\begin{bmatrix} l_1 & l_2 & [L_1L_2] \\ l'_1, l''_1 & l'_2, l''_2 & \gamma[L'_1L'_2][L''_1L''_2] \end{bmatrix}_{(n:n'n'')} = (-1)^{\phi}
$$
$$
\times \left[\left\langle \begin{matrix} \frac{-2L'_2-n'}{4} & \frac{-2L'_1-n'}{4} \\ \frac{-2l'_1-n'}{4}, & \frac{-2l'_2-n'}{4} \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{-2L''_2-n''}{4} & \frac{-2L''_1-n''}{4} \\ \frac{-2l''_1-n''}{4}, & \frac{-2l''_2-n''}{4} \end{matrix} \right\rangle \left\langle \begin{matrix} \frac{-2L_2-n}{4} & \frac{-2L_1-n}{4} \\ \frac{-2l_1-n}{4}, & \frac{-2l_2-n}{4} \end{matrix} \right\rangle^{\gamma} \right], \tag{1}
$$

i.e. they coincide, up to phase factor $(-1)^{\phi}$, with the isofactors for the non-compact complementary group [11, 12, 13] chain Sp(4,R)⊃Sp(2,R)×Sp(2,R) in the case for the discrete series of irreps. Particularly, in a special multiplicity-free case (for $L_2 = L_2' = L_2'' = 0$, when the multiplicity label γ is absent), the isofactors of SO(n)⊃SO($n-1$) correspond to the semistretched isofactors of the second kind [14] of Sp(4)⊃SU(2)×SU(2) (see also [15, 16]).

However, neither expressions derived by means of direct integration [2, 5], nor the expressions derived by the re-expansion of the states of the group chains [6, 10] reveal the apparent triangle conditions of the 3j-symbols in these triple-sum series. Only the substitution group technique of the Sp(4) or SO(5) group [17], used together with an analytical continuation procedure, enabled the transformations of the initial triple-sum expressions of [6] into other forms [6, 10, 15, 18], more convenient in the cases close to the stretched ones (e.g. for small values of shift $l_1 + l_2 - l_3$, where $l_3 = L_1$) and turning into the double sums for the canonical basis, the SO(n)⊃SO($n-2$)×SO(2) chain and other cases with specified parameters $l_1'' + l_2'' - l_3'' = 0$ (where $l_3'' = L_1''$). More specified isofactors of SO(n)⊃SO($n-1$) [16] are related to $6j$ coefficients of SU(2) (with some parameters being quartervalued, i.e. multiple of 1/4, in the case when n is odd).

Unfortunately, the empirical phase choices of isofactors in early publications [6, 10, 15, 16] were not correlated with the basis states (cf. [1, 4, 19, 20]) in terms of the Gegenbauer and the Jacobi polynomials and some aspects of the isofactor symmetry problem were left untouched (including the sign change for irreps m of the SO(2) subgroups not revealed also in [1, 4, 19] for the states of SO(3)⊃SO(2) and SO(n)⊃SO($n-2$)×SO(2)). Besides some indefiniteness of the double factorials in the numerator or denominator must be eliminated.

Recently the author has returned to the problem. The unambiguous proof of the most preferable and consistent expressions for the $3j$-symbols of the orthogonal SO(n) and unitary U(n) groups for decomposition of the factorized ultraspherical and polyspherical harmonics (i.e. for the coupling of three most degenerate irreps into scalar representation in the cases of the canonical and semi-canonical bases) was reconsidered in [18] (cf. also [2, 3, 4, 5, 6, 16]), together with a comprehensive review of some adjusted previous results [5, 6, 14, 15], taking into account that some references [2, 3, 4, 6, 15] may be not easily accessible nor free from misprints.

However, the main goal of [18] and this paper is a strict *ab initio* rearrangement of the most symmetric (although banal) finite triple-sum series of the hypergeometric-type in the expressions of the definite integrals involving triplets of the multiplied Gegenbauer and the Jacobi polynomials into less symmetric but more convenient triple (4f), double (11), or single (13b) sum series with the summation intervals depending on the triangular conditions of the corresponding $3j$-symbols. The related triple-hypergeometric series, appearing in the expressions [14] for the semistretched isoscalar factors of the second kind of the chain Sp(4)⊃SU(2)×SU(2), were considered in section 2 of [18], together with their *ab initio* rearrangement using the different expressions [21] for the stretched $9j$ coefficients of SU(2). (These triple sum series may be treated as extensions of the double-hypergeometric series of Kampé de Fériet [22, 23] type, e.g. considered by Lievens and Van der Jeugt [24].)

The well-known special integral involving triplet of the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ [25, 26, 27] in terms of the Clebsch–Gordan coefficients of SU(2)

$$
\begin{aligned}
&\frac{1}{2}\int_{-1}^{1} \mathrm{d}x \left(\tfrac{1+x}{2}\right)^{(\beta_1+\beta_2+\beta_3)/2} \left(\tfrac{1-x}{2}\right)^{(\alpha_1+\alpha_2+\alpha_3)/2} \prod_{a=1}^{3} P_{k_a}^{(\alpha_a,\beta_a)}(x) \\
&= \left[\frac{1}{2l_3+1}\prod_{a=1}^{3}\frac{(k_a+\alpha_a)!(k_a+\beta_a)!}{k_a!(k_a+\alpha_a+\beta_a)!}\right]^{1/2} C^{l_1 l_2 l_3}_{m_1 m_2 m_3} C^{l_1 l_2 l_3}_{n_1 n_2 n_3},
\end{aligned}
\tag{2}
$$

may be derived within the frames of the angular momentum theory [28, 29, 30], when

$$
l_a = k_a + \tfrac{1}{2}(\alpha_a+\beta_a), \quad m_a = \tfrac{1}{2}(\alpha_a+\beta_a), \quad n_a = \tfrac{1}{2}(\beta_a-\alpha_a)
$$

and

$$
\begin{gathered}
\alpha_a = m_a - n_a, \quad \beta_a = m_a + n_a, \quad k_a = l_a - m_a; \\
\alpha_3 = \alpha_1 + \alpha_2, \quad \beta_3 = \beta_1 + \beta_2
\end{gathered}
$$

are non-negative integers (cf. [1]). Unfortunately, quite an elaborate expansion [3, 4] of two multiplied Jacobi or Gegenbauer polynomials in terms of the third Jacobi or Gegenbauer polynomial in frames of (2) gives rather complicated multiple-sum expressions for the integrals involving the ultraspherical or polyspherical functions in the generic SO(n) or U(n) case.

In section 2, the definite integrals involving products of three unrestrained Jacobi polynomials are initially expressed using the direct (cf. [2, 5]) integration procedure as different (more or less symmetric) triple-sums in terms of beta and gamma functions. Later they are rearranged straightforwardly (without any allusion to special isofactors of Sp(4)) to a more convenient form, with smaller number of sums, or at least, with a richer structure of the summation intervals (responding to the triangular conditions of the coupling coefficients) and better possibilities of summation (especially, under definite restrictions or for some coinciding parameters). In section 3, these results are extended and specified for the definite integrals involving the triplets of the multiplied Gegenbauer polynomials and related special Jacobi polynomials.

In section 4, some normalization and phase choice peculiarities of the canonical basis states and matrix elements of the symmetric (class-one) irreducible representations of SO(n) are discussed. Then we consider the corresponding expressions of $3j$-symbols and Clebsch–Gordan coefficients of SO(n), factorized in terms of integrals involving triplets of the Gegenbauer polynomials (preferable in comparison with results of [5]) and extreme (summable) $3j$-symbols, together with the alternative phase systems.

In section 5, the semicanonical basis states and matrix elements of the symmetric (class-one) irreducible representations of SO(n) for restriction SO(n) $\supset$ SO(n')$\times$SO(n'') ($n'+n''=n$) are discussed. The corresponding factorized $3j$-symbols and Clebsch–Gordan coefficients, expressed in terms of integrals involving triplets of the Jacobi polynomials and extreme $3j$-symbols, are considered, together with a special approach to the $n''=2$ and $n'=n''$ cases.

The spherical functions for the canonical chain of unitary groups U(n) $\supset$ U($n-1$)$\times$U(1)$\supset$ $\cdots$ $\supset$U(2)$\times$U(1) $\supset$U(1) correspond to the matrix elements of the class-two (mixed tensor) representations of U(n), which include the scalar of subgroup U($n-1$) (see [4]). The factorized $3j$-symbols of U(n), related in this case to isofactors of SO($2n$)$\supset$SO($2n-2$)$\times$SO(2), are expressed in section 6, in terms of special integrals involving triplets of the Jacobi polynomials.

In section 7, we discuss the weight lowering (shift) operators of Sp(4) in its enveloping algebra and (in section 8) the analytical continuation relation of triple-sum series with the semistretched isoscalar factors of the second kind of Sp(4) [14]. Particularly, some special cases of these isoscalar factors turn into the double or single sum series for some coinciding values of parameters.

In section 9, the expansions in terms of the double-sum seed isofactors for the most general isofactors of SO(n) for the coupling of the two symmetric irreps in the canonical basis are considered.

2 Integrals involving triplets of Jacobi polynomials

It is convenient for our purposes to use the symmetry property and the two following expressions for the Jacobi polynomials (cf. (16) of section 10.8 of [25], chapter 22 of [26] and definition 2.5.1 of [27]):

$$P_k^{(\alpha,\beta)}(x) = (-1)^k P_k^{(\beta,\alpha)}(-x) \tag{3a}$$

$$= 2^{-k}\sum_m \frac{(-k-\alpha)_m(-k-\beta)_{k-m}}{m!(k-m)!}(-1)^m(1+x)^m(1-x)^{k-m} \tag{3b}$$

$$= (-1)^k\sum_m \frac{(-k-\alpha)_m(k+\alpha+\beta+1)_{k-m}}{m!(k-m)!}\left(\frac{1-x}{2}\right)^{k-m}, \tag{3c}$$

where $\alpha > -1, \beta > -1$ and

$$(c)_n = \prod_{k=0}^{n-1}(c+k) = \frac{\Gamma(c+n)}{\Gamma(c)}$$

are the Pochhammer symbols.

We introduce the following expressions for the integrals involving the product of three Jacobi polynomials $P_{k_1}^{(\alpha_1,\beta_1)}(x)$, $P_{k_2}^{(\alpha_2,\beta_2)}(x)$ and $P_{k_3}^{(\alpha_3,\beta_3)}(x)$ with a measure dependent on $\alpha_0 > -1, \beta_0 >$

-1 and integers $\alpha_a - \alpha_0 \geq 0$, $\beta_a - \beta_0 \geq 0$ $(a = 1, 2, 3)$:

$$\frac{1}{2}\int\limits_{-1}^{1} \mathrm{d}x \left(\frac{1+x}{2}\right)^{(\beta_1+\beta_2+\beta_3-\beta_0)/2} \left(\frac{1-x}{2}\right)^{(\alpha_1+\alpha_2+\alpha_3-\alpha_0)/2} \prod_{a=1}^{3} P_{k_a}^{(\alpha_a,\beta_a)}(x)$$

$$= \widetilde{\mathcal{I}}\left[\begin{array}{cccc} \alpha_0,\beta_0 & \alpha_1,\beta_1 & \alpha_2,\beta_2 & \alpha_3,\beta_3 \\ & k_1 & k_2 & k_3 \end{array}\right] \tag{4a}$$

$$= (-1)^{k_1+k_2+k_3}\widetilde{\mathcal{I}}\left[\begin{array}{cccc} \beta_0,\alpha_0 & \beta_1,\alpha_1 & \beta_2,\alpha_2 & \beta_3,\alpha_3 \\ & k_1 & k_2 & k_3 \end{array}\right] \tag{4b}$$

$$= \sum_{z_1,z_2,z_3} \mathrm{B}\left(1-\tfrac{1}{2}\beta_0+\sum_{a=1}^{3}(\tfrac{1}{2}\beta_a+z_a), 1-\tfrac{1}{2}\alpha_0+\sum_{a=1}^{3}(\tfrac{1}{2}\alpha_a+k_a-z_a)\right)$$
$$\times \prod_{a=1}^{3} \frac{(-1)^{z_a}(-k_a-\alpha_a)_{z_a}(-k_a-\beta_a)_{k_a-z_a}}{z_a!(k_a-z_a)!} \tag{4c}$$

$$= \sum_{z_1,z_2,z_3} \mathrm{B}\left(1-\tfrac{1}{2}\beta_0+\tfrac{1}{2}\sum_{a=1}^{3}\beta_a, 1-\tfrac{1}{2}\alpha_0+\sum_{a=1}^{3}(\tfrac{1}{2}\alpha_a+k_a-z_a)\right)$$
$$\times \prod_{a=1}^{3} \frac{(-1)^{k_a}(-k_a-\alpha_a)_{z_a}(k_a+\alpha_a+\beta_a+1)_{k_a-z_a}}{z_a!(k_a-z_a)!} \tag{4d}$$

$$= \sum_{z_1,z_2,z_3} \frac{(-1)^{k_1+k_3-z_3}(-k_1-\alpha_1)_{z_1}(k_1+\alpha_1+\beta_1+1)_{k_1-z_1}}{z_1!(k_1-z_1)!}$$
$$\times \frac{(-k_2-\beta_2)_{z_2}(k_2+\alpha_2+\beta_2+1)_{k_2-z_2}(-k_3-\alpha_3)_{k_3-z_a}(-k_3-\beta_3)_{z_3}}{z_2!(k_2-z_2)!z_3!(k_3-z_3)!}$$
$$\times \mathrm{B}\left(1-\tfrac{1}{2}\beta_0+\tfrac{1}{2}\sum_{a=1}^{3}\beta_a+\sum_{b=2}^{3}(k_b-z_b), 1-\tfrac{1}{2}\alpha_0+\tfrac{1}{2}\sum_{a=1}^{3}\alpha_a+k_1-z_1+z_3\right) \tag{4e}$$

$$= \mathrm{B}(k_i+\alpha_i+1, k_i+\beta_i+1) \sum_{z_1,z_2,z_3} (-1)^{p_i-p_i''+z_1+z_2+z_3}$$
$$\times \binom{p_i''}{p_i-z_1-z_2-z_3} \frac{(k_i+1)_{z_i}(k_i+\beta_i+1)_{z_i}}{z_i!(2k_i+\alpha_i+\beta_i+2)_{z_i}}$$
$$\times \prod_{a=j,k;a\neq i} \frac{(-k_a-\beta_a)_{z_a}(k_a+\alpha_a+\beta_a+1)_{k_a-z_a}}{z_a!(k_a-z_a)!}, \tag{4f}$$

where the linear combinations (triangular conditions)

$$p_i' = \tfrac{1}{2}(\beta_j+\beta_k-\beta_i-\beta_0) \geq 0, \quad p_i'' = \tfrac{1}{2}(\alpha_j+\alpha_k-\alpha_i-\alpha_0) \geq 0,$$
$$p_i = k_j+k_k-k_i+p_i'+p_i'' \geq 0 \qquad (i,j,k = 1,2,3)$$

and arguments of the binomial coefficients are the non-negative integers. These integrals would otherwise vanish. Three first expressions (4c), (4d) and (4e) (including $(k_1+1)(k_2+1)(k_3+1)$ terms each) are derived directly using expressions (3b), (3c) and their combination, respectively, for the Jacobi polynomials and definite integrals (see equation (6.2.1) of [26] or section 1.1 of [27]) in terms of beta functions $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Two first expressions (4c) and (4d) are invariant under permutations of the sets k_a, α_a, β_a, $a = 1, 2, 3$, but only (4c) satisfies relation (4a)–(4b). Expression (4e) with minimal symmetry[1] is derived using (3c) for $P_{k_1}^{(\alpha_1,\beta_1)}(x)$, (3c) together with (3a) for $P_{k_2}^{(\alpha_2,\beta_2)}(x)$ and (3b) together with (3a) for $P_{k_3}^{(\alpha_3,\beta_3)}(x)$. However, the vanishing of integrals (4a) under spoiled triangular conditions is seen directly only in the final expression (4f), which cannot be derived in a similar manner as (4c)–(4e), but it has been proved in [18] after an elaborated analytical continuation procedure. The ith expression (4f) is invariant under permutations of the sets k_a, α_a, β_a, where $a = 1, 2, 3$; $a \neq i$.

We get a simple rearrangement of (4d) into (4f), after we apply the symmetry relation (4a)–(4b) to (4d) (i.e. interchange α_a and β_a, $a = 0, 1, 2, 3$). When α_0 and β_0 are integers, the ${}_3F_2(1)$ type

[1]It gets the phase factor $(-1)^{k_1+k_2+k_3}$ after interchange of the sets $k_1, \alpha_1, \beta_1, \alpha_3, \alpha_0$ and $k_2, \beta_2, \alpha_2, \beta_3, \beta_0$.

sums over z_i in modified expressions (4d) and (4f) correspond to the CG coefficients of SU(2) with the equivalent Regge [31] 3×3 symbols

$$\left\| \begin{array}{ccc} k_i & k_i+\alpha_i+\beta_i & p_i-z_j-z_k \\ p'_i+\beta_i+k_j+k_k-z_j-z_k & p''_i & k_i+\alpha_i \\ p''_i+\alpha_i & p'_i+k_j+k_k-z_j-z_k & k_i+\beta_i \end{array} \right\| \tag{5a}$$

and

$$\left\| \begin{array}{ccc} p_i-z_j-z_k & k_i & k_i+\alpha_i+\beta_i \\ k_i+\beta_i & p''_i+\alpha_i & p'_i+k_j+k_k-z_j-z_k \\ k_i+\alpha_i & p'_i+\beta_i+k_j+k_k-z_j-z_k & p''_3 \end{array} \right\|, \tag{5b}$$

expressed in the both cases by means of (15.1c) of Jucys and Bandzaitis [28] [see also (7) of section 8.2 of [29]], but with hidden triangular conditions in the first case. For possible non-integer values of α_0 and/or β_0, the doubts as to the equivalence of these finite ${}_3F_2(1)$ series may be caused by the absence of mutually coinciding integer parameters [k_i in (4d) and $\min(p_i - z_j - z_k, p''_i)$ as triangular conditions in (4f), respectively] restricting summation over z_i, unless equation (15.1d) of [28] (together with possible inversion of summation) is used for the CG coefficient of SU(2) with Regge symbol (5b).[2]

In order to demonstrate an analogy with the proof of identity (2.6a)–(2.6c) of [18], we may also consider rearrangement of (4e) into (4f) with the specified $i = 3$. For special values of parameters $k_1 = 0$, $p_3 = k_2 - k_3 + p'_3 + p''_3$, the ${}_3F_2(1)$ type sums over z_3 in (4e) and (4f) correspond to the equivalent Regge symbols related to different expressions (15.1b) and (15.1c) of [28] or (4) and (7) of section 8.2 of [29] for the CG coefficients of SU(2). Hence, for $k_1 = z_1 = 0$ the double-sum expressions (4e) and (4f) are equal. This identity may be extended applying the symmetry relation (4a)–(4b) independently to the left-hand or to the right-hand side. Further, we see that the double-sum over z_2 and z_3 in the general triple-sum expression (4e) corresponds to its $k_1 = z_1 = 0$ version with parameters $\frac{1}{2}\alpha_1, p''_3$ and p_3 replaced by $\frac{1}{2}\alpha_1 + k_1 - z_1, p''_3 + k_1 - z_1$ and $p_3 - z_1$, respectively. Now it is expedient to insert the double-sum version of (4e)–(4f) with interchanged α_a and β_a in the right-hand side and corresponding parameters replaced by $\frac{1}{2}\alpha_1 + k_1 - z_1, p''_3 + k_1 - z_1$ and $p_3 - z_1$. Hence, we derive the final result, which is related to (4f) with respect to the symmetry relation (4a)–(4b). Note that direct transformation of single ${}_3F_2(1)$ series [32, 33, 34] is useless in this case.

An advantage of our new expression (4f) is the restriction of all three summation parameters $z_1 + z_2 + z_3$ by the triangular condition p_i, in contrast with remaining expressions. Alternatively, the linear combination of the summation parameters $p_i - z_1 - z_2 - z_3 \geq 0$ is restricted in addition by p''_i (or by p'_i, if symmetry relation (4a)–(4b) is applied) only in the ith version of (4f). Thus, there are three cases when the expressions for integral $\widetilde{\mathcal{I}}\,[\cdots]$ are completely summable and six cases when they turn into double sums, in addition to the double sums which appear for $k_a = 0$ ($a = 1, 2, 3$). However, the factorization of (4c)–(4f) for $\alpha_0 = \beta_0 = 0$, $\alpha_i = \alpha_j + \alpha_k$, $\beta_i = \beta_j + \beta_k$ into a product of two CG coefficients of SU(2) is not straightforward to prove.

For $\alpha_0 = 0$ and $\alpha_3 = \alpha_1 + \alpha_2$, the integrals involving the product of three Jacobi polynomials (4f) turn into the double sums (6a) or (6b)

$$\begin{aligned} \widetilde{\mathcal{I}} \left[\begin{array}{cccc} 0, \beta_0 & \alpha_1, \beta_1 & \alpha_2, \beta_2 & \alpha_1+\alpha_2, \beta_3 \\ & k_1 & k_2 & k_3 \end{array} \right] &= \mathrm{B}(k_3+\alpha_1+\alpha_2+1, k_3+\beta_3+1) \\ &\times \sum_{z_1, z_2} \frac{(k_3+1)_{p_3-z_1-z_2} (k_3+\beta_3+1)_{p_3-z_1-z_2}}{(p_3-z_1-z_2)!(2k_3+\alpha_1+\alpha_2+\beta_3+2)_{p_3-z_1-z_2}} \\ &\times \prod_{a=1}^{2} \frac{(-k_a-\beta_a)_{z_a}(k_a+\alpha_a+\beta_a+1)_{k_a-z_a}}{z_a!(k_a-z_a)!} \\ &= \mathrm{B}\left(k_1+\alpha_1+1, k_1+\beta_1+1\right) \\ \times \sum_{z_2, z_3} \binom{k_2+\alpha_2}{z_2} & \frac{(-1)^{\alpha_2-z_2}(k_2+\alpha_2+\beta_2+1-z_2)_{k_2}(-k_3-\beta_3)_{z_3}}{z_3!(k_3-z_3)!k_2!} \end{aligned} \tag{6a}$$

[2]The proof of relation between the corresponding finite ${}_3F_2(1)$ series in (4d) and (4f) based on composition of Thomae's transformation formulae (see [27, 32, 33]) or their Whipple's specifications for single restricting parameter (see [32, 34]) is rather complicated.

$$\times\frac{(k_3+\alpha_3+\beta_3+1)_{k_3-z_3}(k_1+1)_{p_1-z_2-z_3}(k_1+\beta_1+1)_{p_1-z_2-z_3}}{(p_1-z_2-z_3)!(2k_1+\alpha_1+\beta_1+2)_{p_1-z_2-z_3}}, \tag{6b}$$

both related to the Kampé de Fériet [22, 23] functions $F^{2:2}_{2:1}$. It is evident that the triple series (4c) and (4d) with $\alpha_0=0$ may be also extended to the negative integer values of α_2,

$$\begin{aligned}&\widetilde{\mathcal{I}}\left[\begin{array}{cccc} 0,\beta_0 & \alpha_1,\beta_1 & \alpha_2,\beta_2 & \alpha_3,\beta_3 \\ & k_1 & k_2 & k_3\end{array}\right]\\ &=(-1)^{\alpha_2}\frac{(k_2+\alpha_2)!(k_2+\beta_2)!}{k_2!(k_2+\alpha_2+\beta_2)!}\widetilde{\mathcal{I}}\left[\begin{array}{cccc} 0,\beta_0 & \alpha_1,\beta_1 & -\alpha_2,\beta_2 & \alpha_3,\beta_3 \\ & k_1 & k_2+\alpha_2 & k_3\end{array}\right],\end{aligned} \tag{7}$$

with invariant values of p_1 and p_3. Hence, using (6a) for the right-hand side of (7), the left-hand side of (7) may be expressed as the double sum for $\alpha_3=\alpha_1-\alpha_2$ and (6b) may be derived after interchange of k_1,α_1,β_1 and k_3,α_3,β_3.

3 Integrals involving triplets of Gegenbauer polynomials

The Gegenbauer (ultraspherical) polynomial $C^{\lambda}_k(\cos\theta)$ may be expressed as the finite series [25, 26], or in terms of the special Jacobi polynomial (cf. [25, 27])

$$C^{\lambda}_k(\cos\theta)=\sum_{m=0}^{[k/2]}\frac{(-1)^m(\lambda)_{k-m}}{m!(k-2m)!}2^{k-2m}\cos^{k-2m}\theta \tag{8a}$$

$$=\frac{(2\lambda)_k}{(\lambda+1/2)_k}P_k^{(\lambda-1/2,\lambda-1/2)}(\cos\theta), \tag{8b}$$

where $[k/2]=\frac{1}{2}(k-\delta)$ ($\delta=0$ or 1) is an integer part of $k/2$ and (8b) includes almost twice as many terms as (8a).

Now we may express the integrals involving the product of three Gegenbauer polynomials $C^{\lambda_i}_{k_i}(x)$ as follows:

$$\int_0^{\pi}\mathrm{d}\theta(\sin\theta)^{\lambda_1+\lambda_2+\lambda_3-n/2+1}\prod_{i=1}^{3}C^{\lambda_i}_{k_i}(\cos\theta)$$

$$=\sum_{z_1,z_2,z_3}\mathrm{B}\left(1+\tfrac{1}{2}\left(\sum_{a=1}^{3}\lambda_a-n/2\right),\tfrac{1}{2}+\sum_{a=1}^{3}(\tfrac{1}{2}k_a-z_a)\right)$$

$$\times\prod_{i=1}^{3}\frac{(-1)^{z_i}2^{k_i-2z_i}(\lambda_i)_{k_i-z_i}}{z_i!(k_i-2z_i)!} \tag{9a}$$

$$=(-1)^{[k_1/2]+[k_2/2]+[k_3/2]}\prod_{a=1}^{3}\frac{(\lambda_a)_{(k_a+\delta_a)/2}}{(1/2)_{(k_a+\delta_a)/2}}$$

$$\times\widetilde{\mathcal{I}}\left[\begin{array}{cccc} -\frac{1}{2},\frac{n-3}{2} & \delta_1-\frac{1}{2},\lambda_1-\frac{1}{2} & \delta_2-\frac{1}{2},\lambda_2-\frac{1}{2} & \delta_3-\frac{1}{2},\lambda_3-\frac{1}{2} \\ & \frac{1}{2}(k_1-\delta_1) & \frac{1}{2}(k_2-\delta_2) & \frac{1}{2}(k_3-\delta_3)\end{array}\right] \tag{9b}$$

$$=(-1)^{p_i'/2}\mathrm{B}\left(\tfrac{1}{2}(k_i+\delta_i+1),\lambda_i+\tfrac{1}{2}(k_i-\delta_i+1)\right)$$

$$\times\prod_{a=1}^{3}\frac{(\lambda_a)_{(k_a+\delta_a)/2}}{(1/2)_{(k_a+\delta_a)/2}}\sum_{z_1,z_2,z_3}\binom{(\delta_j+\delta_k-\delta_i)/2}{p_i/2-z_1-z_2-z_3}$$

$$\times(-1)^{z_1+z_2+z_3}\frac{((k_i-\delta_i)/2+1)_{z_i}\,(\lambda_i+(k_i-\delta_i+1)/2)_{z_i}}{z_i!(k_i+\lambda_i+1)_{z_i}}$$

$$\times\prod_{a\neq i}\frac{(-\lambda_a-(k_a-\delta_a-1)/2)_{z_a}\,(\lambda_a+(k_a+\delta_a)/2)_{(k_a-\delta_a)/2-z_a}}{z_a!\,((k_a-\delta_a)/2-z_a)!} \tag{9c}$$

$$=\widetilde{\mathcal{I}}\left[\begin{array}{cccc} \overline{\alpha}_0,\overline{\alpha}_0 & l_1'+\overline{\alpha}_0,l_1'+\overline{\alpha}_0 & l_2'+\overline{\alpha}_0,l_2'+\overline{\alpha}_0 & l_3'+\overline{\alpha}_0,l_3'+\overline{\alpha}_0 \\ & k_1 & k_2 & k_3\end{array}\right]$$

$$\times 2^{l_1'+l_2'+l_3'+n-2}\prod_{i=1}^{3}\frac{(2\lambda_i)_{k_i}}{(\lambda_i+1/2)_{k_i}}, \tag{9d}$$

where in (9c)

$$p_i' = \tfrac{1}{2}(\lambda_j+\lambda_k-\lambda_i-\tfrac{n}{2}+1)\geq 0, \quad p_i'' = \tfrac{1}{2}(\delta_j+\delta_k-\delta_i),$$
$$p_i = \tfrac{1}{2}(k_j+k_k-k_i)+p_i'\geq 0$$

and in (9d) $l_a' = \lambda_a - \frac{n}{2}+1$, $\overline{\alpha}_0 = \frac{n-3}{2}$. In accordance with (9c), these integrals would otherwise vanish. Expressions (9a) (cf. [5]) and (9d) (cf. [2]) are derived directly (using definite integrals (6.2.1) of [26] or [27] in terms of beta functions). Further (9a) is recognized as consistent with a particular case of (4d)[3] denoted by (9b) and re-expressed, in accordance with (4d)–(4f), in the most convenient form as (9c), where $\delta_1,\delta_2,\delta_3 = 0$ or 1 (in fact either $\delta_1=\delta_2=\delta_3=0$, or $\delta_a=\delta_b=1$, $\delta_c=0$) and $\frac{1}{2}(k_a-\delta_a)$ $(a=1,2,3)$ are integers.

Expression (9a) includes $\frac{1}{8}\prod_{a=1}^{3}(k_a-\delta_a+2)$ terms, when (9d), used together with (4c) or (4d), each includes $\prod_{a=1}^{3}(k_a+1)$ terms; otherwise, the number of terms in the ith version of the most convenient formula (9c) never exceeds

$$A_i = (p_i''+1)\min\left[\tfrac{1}{2}(p_i+1)(p_i-p_i''+2),\tfrac{1}{4}\prod_{a\neq i}(k_a-\delta_a+2)\right], \tag{10}$$

where a set i,j,k is a transposition of 1,2,3. The number of terms decreases in comparison with (10) in the intermediate region

$$\tfrac{1}{2}\min(k_j-\delta_j,k_k-\delta_k) < p_i < \tfrac{1}{2}(k_j-\delta_j+k_k-\delta_k).$$

Actually, expression (9c) is related to the Kampé de Fériet [22, 23] function $F^{2:2}_{2:1}$ (for $p_i''=0$) or to the sum of two such functions (when $p_i''=1$). Hence, after comparing three different versions of (9c), the rearrangement formulae of special Kampé de Fériet functions $F^{2:2}_{2:1}$ can be derived.

Finally we may express the integrals involving the product of three Gegenbauer polynomials $C^{l_1'+n/2-1}_{l_1-l_1'}(x)$, $C^{l_2'+n/2-1}_{l_2-l_2'}(x)$ and $C^{l_3'+n/2-1}_{l_3-l_3'}(x)$, needed in the next section as follows:

$$\begin{aligned}
&\int_0^{\pi} d\theta(\sin\theta)^{l_1'+l_2'+l_3'+n-2}\prod_{i=1}^{3}C^{l_i'+n/2-1}_{l_i-l_i'}(\cos\theta)\\
&= (-1)^{(l_j'+l_k'-l_i')/2}\mathrm{B}\left(\tfrac{1}{2}(l_i-l_i'+\delta_i+1),\tfrac{1}{2}(l_i+l_i'-\delta_i+n-1)\right)\\
&\times\prod_{a=1}^{3}\frac{(l_a'+n/2-1)_{(l_a-l_a'+\delta_a)/2}}{(1/2)_{(l_a-l_a'+\delta_a)/2}}\sum_{z_1,z_2,z_3}\binom{(\delta_j+\delta_k-\delta_i)/2}{(l_j+l_k-l_i)/2-z_1-z_2-z_3}\\
&\times\prod_{a\neq i}\frac{(-(l_a+l_a'-\delta_a+n-3)/2)_{z_a}\,((l_a+l_a'+\delta_a+n)/2-1)_{(l_a-l_a'-\delta_a)/2-z_a}}{z_a!\,((l_a-l_a'-\delta_a)/2-z_a)!}\\
&\times(-1)^{z_1+z_2+z_3}\frac{((l_i-l_i'-\delta_i)/2+1)_{z_i}\,((l_i+l_i'-\delta_i+n-1)/2)_{z_i}}{z_i!(l_i+n/2)_{z_i}},
\end{aligned} \tag{11}$$

where $p_i=\frac{1}{2}(l_j+l_k-l_i)\geq 0$ and $p_i'=\frac{1}{2}(l_j'+l_k'-l_i')\geq 0$ are integers.

Now we consider more specified integrals involving several Gegenbauer polynomials. At first, using (11) with $i=3$ and $z_2=\delta_2=0$, $z_3=\frac{1}{2}(l_1+l'-l_3)-z_1$, we take special integral involving two multiplied Gegenbauer polynomials (where third trivial polynomial $C_0^{l'+n/2-1}(x)=1$ may be inserted) in terms of the summable balanced (Saalschützian) ${}_3F_2(1)$ series (cf. [32, 33]) and write:

$$\int_0^{\pi}(\sin\theta)^{2l'+n-2}C^{l'+n/2-1}_{l_1-l'}(\cos\theta)C^{l'+n/2-1}_{l'-l'}(\cos\theta)C^{n/2-1}_{l_3}(\cos\theta)d\theta$$

[3] This was the reason why (4d) has been introduced in [18].

$$= \int_0^\pi (\sin\theta)^{2l'+n-2} C_{l_1-l'}^{l'+n/2-1}(\cos\theta) C_{l_3}^{n/2-1}(\cos\theta)\mathrm{d}\theta \tag{12a}$$

$$= \frac{(-1)^{(l_3-l_1+l')/2}\pi\, l')!(l_1+l'+n-3)!}{2^{2l'+n-3}(l_1-l')!(J'-l_1)!(J'-l_3)!\,\Gamma(n/2-1)} \times \frac{\Gamma(J'-l'+n/2-1)}{\Gamma(l'+n/2-1)\Gamma(J'+n/2)}, \tag{12b}$$

where $J' = \frac{1}{2}(l_1+l'+l_3)$.

Using the expansion formula (linearization Theorem 6.8.2 of [27]) of two multiplied Gegenbauer polynomials (zonal spherical functions) $C_l^p(x)C_k^p(x)$ in terms of the third polynomial $C_n^p(x)$ (cf. [1]), where $l+k-n$ is even, the special integral involving three Gegenbauer polynomials (with coinciding superscripts in two cases) may be also expanded in terms of integrals (12b) and may be presented as follows:

$$\int_0^\pi (\sin\theta)^{2l'+n-2} C_{l_1-l'}^{l'+n/2-1}(\cos\theta) C_{l_2-l'}^{l'+n/2-1}(\cos\theta) C_{l_3}^{n/2-1}(\cos\theta)\mathrm{d}\theta$$

$$= \frac{\pi l'!}{2^{2l'+n-3}\Gamma^3(n/2-1)\Gamma(l'+n/2-1)} \times \sum_{k=|l_1-l_2|+l'}^{l_1+l_2-l'} \frac{(-1)^{(l_3+l'-k)/2}(k+n/2-1)}{\nabla^2\left(l'/2, l_3/2+n/4-1, k/2+n/4-1\right)} \times \frac{\nabla^2\left((l_1+l'+n)/2-2, l_2/2+n/4-1, k/2+n/4-1\right)}{\nabla^2\left((l_1-l')/2, l_2/2+n/4-1, k/2+n/4-1\right)} \tag{13a}$$

$$= \frac{\pi\, l'! \prod_{a=1}^3 \Gamma(J-l_a+n/2-1)}{2^{2l'+n-3}\Gamma(n/2-1)\Gamma(l'+n/2-1)\Gamma(J+n/2)} \times \sum_u \frac{(-1)^u(J+l'+n-3-u)!}{u!(l'-u)!(J-l_1-u)!(J-l_2-u)!(J-l_3-l'+u)!} \times[\Gamma(n/2-1+u)\Gamma(l'+n/2-1-u)]^{-1}, \tag{13b}$$

where $J = \frac{1}{2}(l_1+l_2+l_3)$ and the gamma functions under summation sign in the intermediate formula (13a) (which is equivalent to (15) of [35]) are included into the asymmetric triangle coefficients

$$\nabla(abc) = \left[\frac{(a+b-c)!(a-b+c)!(a+b+c+1)!}{(b+c-a)!}\right]^{1/2} \tag{14a}$$

$$= \left[\frac{\Gamma(a+b-c+1)\Gamma(a-b+c+1)\Gamma(a+b+c+2)}{\Gamma(b+c-a+1)}\right]^{1/2}. \tag{14b}$$

Finally, the sum in (13a) corresponds to a very well-poised ${}_7F_6(1)$ hypergeometric series (which may be rearranged using Whipple's transformation of [27] or (6.10) of [36] into balanced terminating ${}_4F_3(1)$ hypergeometric series) or to the usual $6j$ coefficient of SU(2)

$$\begin{Bmatrix} l'+\frac{1}{2}n-2 & \frac{1}{2}(l_1+n)-2 & \frac{1}{2}l_1 \\ \frac{1}{2}l_3+\frac{1}{4}n-1 & \frac{1}{2}l_2+\frac{1}{4}n-1 & \frac{1}{2}l_2+\frac{1}{4}n-1 \end{Bmatrix} \tag{15}$$

with standard (integer or half-integer) parameters (for n even), in accordance with expression (C3) of the $6j$ coefficient [37] in terms of (14a). For n odd some of its parameters may be quartervalued (i.e. multiple of 1/4). Using the most symmetric (Racah) expression [28, 29] for (15), the final expression (13b) with single sum is derived. Intervals of summation are restricted by $\min(l', J-l_1, J-l_2, J-l_3)$ and, of course, (13b) coincide with result of Vilenkin [1] for $l'=0$. Nevertheless, the use of less symmetric expressions (29.1b) and (29.1c) of Jucys and Bandzaitis [28] (see also (5) and (6) in section 9.2 of [29]) for $6j$ coefficients (15), together with Dougall's summation formula [32] of the very well-poised series, allowed us [38] to derive expressions of $6j$-symbols for symmetric representations of SO(n) as the double series.

Comparing expansion (9c) of the integrals involving triplets of the Gegenbauer polynomials with (9d), we may write an expression for the integrals involving triplets of special Jacobi polynomials, with mutually equal superscripts,

$$\widetilde{\mathcal{I}}\left[\begin{array}{cccc}\alpha_0,\alpha_0 & \alpha_1,\alpha_1 & \alpha_2,\alpha_2 & \alpha_3,\alpha_3\\ & k_1 & k_2 & k_3\end{array}\right]$$
$$=\frac{[1+(-1)^{p_i}]\,\mathrm{B}(1/2,k_i+\alpha_i+1)}{2^{k_1+k_2+k_3+\alpha_1+\alpha_2+\alpha_3-\alpha_0+2}(1/2)_{(k_j+\delta_j)/2}(1/2)_{(k_k+\delta_k)/2}}$$
$$\times\sum_{z_1,z_2,z_3}(-1)^{p_i'+(k_j+\delta_j+k_k+\delta_k)/2+z_1+z_2+z_3}\binom{(\delta_j+\delta_k-\delta_i)/2}{p_i/2-z_1-z_2-z_3}$$
$$\times\prod_{a=j,k;a\neq i}\frac{(-k_a-\alpha_a)_{(k_a+\delta_a)/2+z_a}\,(\alpha_a+(k_a+\delta_a+1)/2)_{(k_a-\delta_a)/2-z_a}}{z_a!\,((k_a-\delta_a)/2-z_a)!}$$
$$\times\binom{(k_i-\delta_i)/2+z_i}{z_i}\frac{(\alpha_i+(k_i-\delta_i)/2+1)_{z_i}}{(\alpha_i+k_1+3/2)_{z_i}}. \tag{16}$$

Here

$$p_i=k_j+k_k-k_i+p_i'+p_i'',\quad \tfrac{1}{2}(k_i-\delta_i),\quad \delta_i=0\text{ or }1,$$
$$p_i'=p_i''=\tfrac{1}{2}(\alpha_j+\alpha_k-\alpha_i-\alpha_0)\qquad(i,j,k=1,2,3)$$

are non-negative integers.

Comparing expansion (13b) of the integrals involving more specified triplets of the Gegenbauer polynomials with (9d), we may also write an expression for the integrals involving triplets of special Jacobi polynomials,

$$\widetilde{\mathcal{I}}\left[\begin{array}{cccc}\alpha_0,\alpha_0 & \alpha_1,\alpha_1 & \alpha_1,\alpha_1 & \alpha_0,\alpha_0\\ & k_1 & k_2 & k_3\end{array}\right]$$
$$=\frac{[1+(-1)^{p_1}]2^{2\alpha_0-2}(\alpha_1-\alpha_0)!\Gamma(\alpha_1+1/2)\prod_{a=1}^{3}\Gamma(p_a/2+\alpha_0+1/2)}{\Gamma(1/2)\Gamma\left((k_1+k_2+k_3)/2+\alpha_1+3/2\right)}$$
$$\times\frac{\Gamma(\alpha_1+1+k_1)\Gamma(\alpha_1+1+k_2)\Gamma(\alpha_0+1+k_3)}{\Gamma(2\alpha_1+1+k_1)\Gamma(2\alpha_1+1+k_2)\Gamma(2\alpha_0+1+k_3)}$$
$$\times\sum_u\frac{(-1)^u\left((k_1+k_2+k_3)/2+2\alpha_1-u\right)!}{u!(\alpha_1-\alpha_0-u)!(p_1/2-u)!(p_2/2-u)!(p_3/2+\alpha_0-\alpha_1+u)!}$$
$$\times[\Gamma(\alpha_0+1/2+u)\Gamma(\alpha_1+1/2-u)]^{-1} \tag{17}$$

in terms of the balanced (Saalschützian) ${}_4F_3(1)$ type series [32, 33]. Here

$$p_i=k_j+k_k-k_i+2p_i',\quad p_1'=p_2'=0,\quad p_3'=\alpha_1-\alpha_0$$

are integers.

4 Canonical basis states and coupling coefficients of SO(n)

The canonical basis states of the symmetric (class-one) irreducible representation $l=l_{(n)}$ for the chain SO(n)$\supset$SO($n-1$) $\supset\cdots\supset$SO(3)$\supset$SO(2) are labelled by the $(n-2)$-tuple $M=(l_{(n-1)},N)=(l_{(n-1)},...,l_{(3)},m_{(2)})$ of integers

$$l_{(n)}\geq l_{(n-1)}\geq...\geq l_{(3)}\geq|m_{(2)}|. \tag{18}$$

The dimension of representation space is

$$d_l^{(n)}=\frac{(2l+n-2)(l+n-3)!}{l!(n-2)!}. \tag{19}$$

Special matrix elements $D^{n,l}_{M0}(g)$ of SO(n) irreducible representation $l_{(n)}=l$ with zero for the $(n-2)$-tuple (0,...,0) depend only on the rotation (Euler) angles $\theta_{n-1},\theta_{n-2},...,\theta_2,\theta_1$ (coordinates on the unit sphere S_{n-1}) and may be factorized as

$$D^{n,l}_{M0}(g)=t^{n,l}_{l'0}(\theta_{n-1})D^{n-1,l'}_{N0}(g'). \tag{20}$$

Here $D^{n-1,l'}_{N0}(g')$ are the matrix elements of SO($n-1$) irrep $l_{(n-1)}=l'$ (with coordinates on the unit sphere S_{n-2}). Special matrix elements of SO(n) ($n>3$) irreducible representation $l_{(n)}=l$ with the SO($n-1$) irrep labels $l_{(n-1)}=l'$ and 0 and SO($n-2$) label $l_{(n-2)}=0$ for rotation with angle θ_{n-1} in the (x_n,x_{n-1}) plane are written in terms of the Gegenbauer polynomials as follows:

$$\begin{aligned}t^{n,l}_{l'0}(\theta_{n-1})=&\left[\frac{l!(l-l')!(n-3)!(l'+n-4)!(2l'+n-3)}{l'!(l+l'+n-3)!(l+n-3)!}\right]^{1/2}\\ &\times(n/2-1)_{l'}2^{l'}\sin^{l'}\theta_{n-1}C^{l'+n/2-1}_{l-l'}(\cos\theta_{n-1}),\end{aligned} \tag{21}$$

(see [1]). Function (21) corresponds to the wavefunction $\Psi^c_{k,l'}(\theta)=\Psi^{l+(n-3)/2}_{l-l',l'}(\theta)$ of the tree technique (of the type 2b, see (2.4) of [19]) with factor

$$\left[\frac{\Gamma((n-1)/2)\sqrt{\pi}\,d^{(n-1)}_{l'}}{\Gamma(n/2)\,d^{(n)}_{l}}\right]^{1/2},$$

for appropriate normalization in the case of integration over the group volume ($0\le\theta\le\pi$) with measure $\mathrm{B}^{-1}\left((n-1)/2,1/2\right)\sin^{n-2}\theta\mathrm{d}\theta$. The remaining Euler angles are equal to 0 for the matrix element (21). In the case of SO(3), we obtain

$$D^{3,l}_{m0}(\theta_2,\theta_1)=(-1)^{(l'-m)/2}t^{3,l}_{l'0}(\theta_2)\mathrm{e}^{\imath m\theta_1},\ l'=|m|, \tag{22}$$

in accordance with the relation [1] between the associated Legendre polynomials $P^m_l(x)$ and special Gegenbauer polynomials $C^{m+1/2}_{l-m}(x)$ and the behavior of $P^m_l(x)$ under the reflection of m.

The corresponding $3j$-symbols for the chain SO(n)$\supset$SO($n-1$) $\supset\cdots\supset$SO(3) $\supset$SO(2) (denoted by brackets with simple subscript n and labelled by sets $M_a=(l'_a,N_a)$) may be factorized as follows:

$$\begin{aligned}&\begin{pmatrix} l_1 & l_2 & l_3\\ M_1 & M_2 & M_3\end{pmatrix}_n\\ &=\begin{pmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}_n^{-1}\int\limits_{\mathrm{SO}(n)}\mathrm{d}gD^{n,l_1}_{M_10}(g)D^{n,l_2}_{M_20}(g)D^{n,l_3}_{M_30}(g) &\text{(23a)}\\ &=\begin{pmatrix} l_1 & l_2 & l_3\\ l'_1 & l'_2 & l'_3\end{pmatrix}_{(n:n-1)}\begin{pmatrix} l'_1 & l'_2 & l'_3\\ N_1 & N_2 & N_3\end{pmatrix}_{n-1}. &\text{(23b)}\end{aligned}$$

Here the isoscalar factors of $3j$-symbol for the restriction SO(n) $\supset$SO($n-1$) are denoted by brackets with composite subscript $(n:n-1)$ and are expressed in terms of integrals (11) involving the triplets of the Gegenbauer polynomials,

$$\begin{aligned}\begin{pmatrix} l_1 & l_2 & l_3\\ l'_1 & l'_2 & l'_3\end{pmatrix}_{(n:n-1)}=&\begin{pmatrix} l_1 & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}_n^{-1}\begin{pmatrix} l'_1 & l'_2 & l'_3\\ 0 & 0 & 0\end{pmatrix}_{n-1}\\ &\times\left[\frac{\Gamma\left((n-1)/2\right)}{\pi^{5/2}\Gamma(n/2)}\right]^{1/2}\prod_{a=1}^{3}\mathcal{N}^{(n:n-1)}_{l_a;l'_a,\delta_a}\left[\frac{d^{(n-1)}_{l'_a}}{d^{(n)}_{l_a}}\right]^{1/2}\\ &\times\int\limits_0^\pi\mathrm{d}\theta(\sin\theta)^{l'_1+l'_2+l'_3+n-2}\prod_{i=1}^{3}C^{l'_i+n/2-1}_{l_i-l'_i}(\cos\theta),\end{aligned} \tag{24}$$

where

$$\mathcal{N}^{(n:n-1)}_{l_a;l'_a,\delta_a}=2^{l'_a+n/2-2}\Gamma(l'_a+n/2-1)\left[\frac{(l_a-l'_a)!(2l_a+n-2)}{(l_a+l'_a+n-3)!}\right]^{1/2} \tag{25}$$

are normalization factors and particular $3j$-symbols

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_n = (-1)^{\psi_n} \frac{1}{\Gamma(n/2)} \left[\frac{(J+n-3)!}{(n-3)!\,\Gamma(J+n/2)} \right.$$

$$\left. \times \prod_{i=1}^{3} \frac{(l_i+n/2-1)\,\Gamma(J-l_i+n/2-1)}{d_{l_i}^{(n)}(J-l_i)!} \right]^{1/2} \tag{26a}$$

$$= (-1)^{\psi_n} \frac{\widetilde{\nabla}^{-1}_{n[0,1,2,3]}(l_1,l_2;l_3,0)}{\Gamma(n/2)[(n-3)!]^{1/2}} \left[\prod_{i=1}^{3} \frac{l_i+n/2-1}{d_{l_i}^{(n)}} \right]^{1/2} \tag{26b}$$

(vanishing for $J = \frac{1}{2}(l_1+l_2+l_3)$ half-integer) are derived in [5] (see also special Clebsch–Gordan coefficients [3, 6, 16]). The triangular coefficients $\widetilde{\nabla}_{n[0,1,2,3]}(\cdots)$ in (26b) are expressed as follows:

$$\widetilde{\nabla}_{n[0,1,2,3]}(a,b;e,f)$$

$$= \left[\frac{\left(\frac{1}{2}(a-b+e-f)\right)!\,\left(\frac{1}{2}(b-a+e-f)\right)!}{\Gamma\left(\frac{1}{2}(a-b+e+f+n)-1\right)\Gamma\left(\frac{1}{2}(b-a+e+f+n)-1\right)} \right.$$

$$\left. \times \frac{\left(\frac{1}{2}(a+b-e-f)\right)!\,\Gamma\left(\frac{1}{2}(a+b+e-f+n)\right)}{\Gamma\left(\frac{1}{2}(a+b-e+f+n)-1\right)\left(\frac{1}{2}(a+b+e+f)+n-3\right)!} \right]^{1/2} \tag{27}$$

and in general

$$\widetilde{\nabla}_{n[i_1\ldots i_k]}(a,b;e,f) = \left(\prod_{i=0}^{7} \widetilde{A}_i \right)^{1/2} \left(\prod_{i\to[i_1\ldots i_k]} \widetilde{A}_i \right)^{-1}, \tag{28}$$

$$\begin{array}{ll} \widetilde{A}_0 = \left(\frac{1}{2}(a+b+e+f)+n-3\right)!, & \widetilde{A}_4 = \Gamma\left(\frac{1}{2}(a+b+e-f+n)\right), \\ \widetilde{A}_1 = \Gamma\left(\frac{1}{2}(b-a+e+f+n)-1\right), & \widetilde{A}_5 = \left(\frac{1}{2}(b-a+e-f)\right)!, \\ \widetilde{A}_2 = \Gamma\left(\frac{1}{2}(a-b+e+f+n)-1\right), & \widetilde{A}_6 = \left(\frac{1}{2}(a-b+e-f)\right)!, \\ \widetilde{A}_3 = \Gamma\left(\frac{1}{2}(a+b-e+f+n)-1\right), & \widetilde{A}_7 = \left(\frac{1}{2}(a+b-e-f)\right)!. \end{array} \tag{29}$$

Equation (24) together with (9a) is equivalent to the result of [5], but its most convenient form is obtained[4] when the special integral is expressed by means of double-sum expression (11) (for $i = 1, 2$ or 3, minimizing (10)), which ensures its finite rational structure for the fixed shift $\frac{1}{2}(l_1+l_2-l_3)$ of parameters. In the case of $l'_i = 0$, expression (13b) for the special integral is more convenient in accordance with [16]. In (26a), $J - l_i$ ($i = 1, 2, 3$) and J are non-negative integers and $\psi_3 = J$, in accordance with the angular momentum theory [28, 29]. We may take also

$$\psi_n = J \tag{30}$$

(see [5]) for $n \geq 4$, in order to obtain the isofactors (24) positive where the maximal values of parameters $l'_1 = l_1, l'_2 = l_2, l'_3 = l_3$.

Only by taking into account the phase factor $(-1)^{(l'-m)/2}$ of (22), we can obtain the consistent signs of the usual Wigner coefficients ($3j$-symbols)

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1 & l'_2 & l'_3 \end{pmatrix}_{(3:2)} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{31}$$

of SO(3) or SU(2) (where $l'_a = |m_a|$ and $m_1 + m_2 + m_3 = 0$), with

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \\ 0 & 0 & 0 \end{pmatrix}_2 = \delta_{\max(l'_1,l'_2,l'_3),(l'_1+l'_2+l'_3)/2} (-1)^{(l'_1+l'_2+l'_3)/2} \tag{32}$$

consequently appearing in (24) for $n = 3$.[5]

[4] Note, that the factors under the square root form the rational numbers, taking into account that $\Gamma(1/2) = \sqrt{\pi}$.

[5] Of course, in this case the usual expressions [1, 28, 29, 30] of the Clebsch–Gordan or Wigner coefficients of SU(2) are more preferable in comparison with equation (24).

We may write (cf. [5]) the following dependence between special Clebsch–Gordan coefficients (denoted by square brackets with subscript) and $3j$-symbols of SO(n):

$$\left[\begin{array}{ccc} l_1 & l_2 & l_3 \\ M_1 & M_2 & M_3 \end{array}\right]_n \left[\begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array}\right]_n$$

$$= d_l^{(n)} \int\limits_{\mathrm{SO}(n)} \mathrm{d}g D^{n,l_1}_{M_1 0}(g) D^{n,l_2}_{M_2 0}(g) \overline{D^{n,l_3}_{M_3 0}(g)} \tag{33a}$$

$$= d_l^{(n)} (-1)^{l_3 - m_3} \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ M_1 & M_2 & \overline{M_3} \end{array}\right)_n \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array}\right)_n, \tag{33b}$$

where the $(n-2)$-tuple $\overline{M_3}$ is obtained from the $(n-2)$-tuple M_3 after reflection of the last parameter m_3. Then in the phase system with $\psi_n = J$, we obtain the following relation for the isofactors of CG coefficients in the canonical basis:

$$\left[\begin{array}{ccc} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{array}\right]_{(n:n-1)} = (-1)^{l_3 - l_3'} \left[\frac{d^{(n)}_{l_3}}{d^{(n-1)}_{l_3'}}\right]^{1/2} \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ l_1' & l_2' & l_3' \end{array}\right)_{(n:n-1)}, \tag{34}$$

which, together with (24), (26a), (30) and (11) or (9b) substituted by (4c), allows us to obtain the expressions for isofactors of SO(n) ⊃SO($n-1$) derived in [6] and satisfying the same phase conditions.

However, as it has been noted in [10], the choice (30) of ψ_n does not give the correct phases for special isofactors of SO(4) [39] in terms of $9j$ coefficients of SU(2) [28] and for isofactors of SO(5)⊃SO(4), as considered in [14, 17, 40]. The contrast of the phases is caused by the fact that the signs of the matrix elements of infinitesimal operators

$$A_{k,k-1} = x_k \frac{\partial}{\partial x_{k-1}} - x_{k-1} \frac{\partial}{\partial x_k}, \quad k = 3, ..., n,$$

(with exception of $A_{2,1}$) between the basis states [1] of SO(n) in terms of Gegenbauer polynomials (in x_k/r_k, $r_k^2 = x_1^2 + ... + x_k^2$ variables) are opposite to the signs of the standard (Gel'fand–Tsetlin) matrix elements [41, 42, 43]. We eliminate this difference of phases and our results match with the isofactors for decomposition of the general and vector irreps $m_n \otimes 1$ of SO(n) [43, 44] (specified also in [45, 46]) after we multiply isofactors of CG coefficients for the restriction SO(n)⊃SO($n-1$) ($n \geq 4$), i.e. the left-hand side of (34), by

$$(-1)^{(l_1 + l_2 - l_3 - l_1' - l_2' + l_3')/2}$$

(cf. [10]), i.e. after we omit the phase factors $(-1)^{\psi_n}$ and $(-1)^{\psi_{n-1}}$ in the both auxiliary $3j$-symbols of (24), as well as $(-1)^{l_3 - l_3'}$ in relation (34), again keeping the isofactors (34) with the maximal values of parameters $l_1' = l_1, l_2' = l_2, l_3' = l_3$ for this restriction positive. In the both phase systems of the factorized SO(n) CG coefficients ($3j$-symbols) the last factors coincide with the usual CG coefficients ($3j$-symbols) of angular momentum theory [28, 29].

5 Semicanonical bases and coupling coefficients of SO(n)

Further, going to the semicanonical basis of the symmetric (class-one) irrep l for the chain SO(n)⊃ SO(n')×SO(n'')⊃SO(n'–1)×SO($n''-1$)⊃ ⋯, we introduce special matrix elements $D^{n:n',n'';l}_{l'M',l''M'';0}(g)$ depending only on the rotation angles $\theta'_{n'-1}, ..., \theta'_1$ and $\theta''_{n''-1}, ...\theta''_1$ of subgroups SO(n') and SO(n'') and the rotation angle θ_c in $(x_n, x_{n'})$ plane, with the second matrix index taken to be zero as the $(n-2)$-tuple (0,...,0) for scalar of SO($n-1$). These matrix elements may be factorized as follows:

$$D^{n:n',n'';l}_{l'M',l''M'';0}(g) = t^{(n)\,l}_{(n')l'0,(n'')l''0;\,(n-1)0}(\theta_c) D^{n',l'}_{M'0}(g') D^{n'',l''}_{M''0}(g''). \tag{35}$$

Instead of the wavefunction $\Psi^{b,a}_{k,l'',l'}(\theta_c) = \Psi^{l''+n''/2-1,l'+n'/2-1}_{(l-l'-l'')/2,l'',l'}(\theta_c)$ (of the type 2c, see (2.6) of [19]) of the tree technique after renormalization with factor

$$\left[\frac{\Gamma(n'/2)\Gamma(n''/2)\, d^{(n')}_{l'} d^{(n'')}_{l''}}{2\,\Gamma(n/2)\, d^{(n)}_l}\right]^{1/2}$$

for the integration over the group volume ($0 \leq \theta_c \leq \pi/2$) with measure

$$2\mathrm{B}^{-1}(n'/2, n''/2) \sin^{n''-1}\theta_c \cos^{n'-1}\theta_c \mathrm{d}\theta_c,$$

we obtain special matrix elements of the SO(n) irreducible representation l in terms of the Jacobi polynomials

$$\begin{aligned} t^{(n)\,l}_{(n')l'0,(n'')l''0;\,(n-1)0}(\theta_c) &= (-1)^{\varphi_{n'n''}} \left[\frac{d^{(n')}_{l'} d^{(n'')}_{l''} \Gamma(n/2)}{d^{(n)}_{l} \Gamma(n'/2)\Gamma(n''/2)} \right]^{1/2} \mathcal{N}^{(n:n',n'')}_{l:l',l''} \\ &\times \sin^{l''}\theta_c \cos^{l'}\theta_c P^{(l''+n''/2-1,\,l'+n'/2-1)}_{(l-l'-l'')/2}(\cos 2\theta_c), \end{aligned} \tag{36}$$

where the left-hand SO(n')×SO(n'') labels are l', l'' ($n'+n''=n$), the left-hand SO($n'-1$)×SO($n''-1$) and right-hand SO($n-1$) labels are 0 for rotation with angle θ_c in ($x_n, x_{n'}$) plane. Here phase $\varphi_{n'n''}=0$, unless $n''=2$, or $n'=2$, when the left-hand side should be replaced, respectively, by $t^{(n)\,l}_{(n-2)l'0,(2)m'';\,(n-1)0}(\theta_c)$ with $l''=|m''|$, or by $t^{(n)\,l}_{(2)m',(n-2)l''0;\,(n-1)0}(\theta_c)$ with $l'=|m'|$ and

$$\varphi_{n'n''} = \tfrac{1}{2}[\delta_{n''2}(l''-m'') + \delta_{n'2}(l'-m')]$$

on the right-hand side and normalization factor

$$\mathcal{N}^{(n:n',n'')}_{l:l',l''} = \left[\frac{(l+n/2-1)\,((l-l'-l'')/2)!\,\Gamma\left((l+l'+l''+n-2)/2\right)}{\Gamma\left((l-l'+l''+n'')/2\right)\Gamma\left((l+l'-l''+n')/2\right)} \right]^{1/2}. \tag{37}$$

The $3j$-symbols for the chain SO(n)⊃SO(n')×SO(n'')⊃SO($n'-1$)×SO($n''-1$)⊃ ⋯, labelled by the sets $M_i = (l'_i, N'_i; l''_i, N''_i)$ may be factorized as follows:

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & l_3 \\ M_1 & M_2 & M_3 \end{pmatrix}_n &= \begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1, l''_1 & l'_2, l''_2 & l'_3, l''_3 \end{pmatrix}_{(n:n'n'')} \\ &\times \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ N'_1 & N'_2 & N'_3 \end{pmatrix}_{n'} \begin{pmatrix} l''_1 & l''_2 & l''_3 \\ N''_1 & N''_2 & N''_3 \end{pmatrix}_{n''}. \end{aligned} \tag{38}$$

Now the SO(n)⊃SO(n')×SO(n'') isofactor of $3j$-symbol is expressed as follows:

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1, l''_1 & l'_2, l''_2 & l'_3, l''_3 \end{pmatrix}_{(n:n'n'')} &= \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_n^{-1} \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ 0 & 0 & 0 \end{pmatrix}_{n'} \\ &\times \begin{pmatrix} l''_1 & l''_2 & l''_3 \\ 0 & 0 & 0 \end{pmatrix}_{n''} \prod_{a=1}^{3} \mathcal{N}^{(n:n',n'')}_{l_a;l'_a,l''_a} \left[\frac{d^{(n')}_{l'_a} d^{(n'')}_{l''_a}}{d^{(n)}_{l_a}} \right]^{1/2} \\ &\times \mathrm{B}^{1/2}(n'/2, n''/2)\, \widetilde{\mathcal{I}} \begin{bmatrix} \alpha_0, \beta_0 & \alpha_1, \beta_1 & \alpha_2, \beta_2 & \alpha_3, \beta_3 \\ & k_1 & k_2 & k_3 \end{bmatrix}, \end{aligned} \tag{39}$$

in terms of auxiliary $3j$-symbols (26a) of the canonical bases [turning into phase factors of the type 32 for $n'=2$ or $n''=2$], normalization factors (37) and the integrals involving the triplets of Jacobi polynomials (4a)–(4f), with parameters

$$\begin{gathered} k_i = \tfrac{1}{2}(l_i - l'_i - l''_i), \quad \alpha_i = l''_i + n''/2 - 1, \quad \beta_i = l'_i + n'/2 - 1, \\ \alpha_0 = n''/2 - 1, \quad \beta_0 = n'/2 - 1 \end{gathered}$$

and

$$\begin{gathered} p'_i = \tfrac{1}{2}(l'_j + l'_k - l'_i), \quad p''_i = \tfrac{1}{2}(l''_j + l''_k - l''_i), \\ p_i = \tfrac{1}{2}(l_j + l_k - l_i) \qquad (i,j,k = 1,2,3). \end{gathered}$$

The number of terms in expansion (4f) of the integrals involving triplets of Jacobi polynomials never exceeds

$$B_i = \min\left(\tfrac{1}{6}(p_i+1)_3, (p''_i+1)(k_j+1)(k_k+1), \tfrac{1}{2}(p''_i+1)(p_i+1)_2\right) \tag{40a}$$

and decreases in the intermediate region (e.g., when $p''_i < p_i + 1$), described by the volume of the obliquely truncated rectangular parallelepiped of $(p''_i + 1) \times (k_j + 1) \times (k_k + 1)$ size.

In particular, in the case of $n'' = 2$ parameters l''_1, l''_2, l''_3 in $3j$-symbol (39) should be replaced by $m''_1 = \pm l''_1, m''_2 = \pm l''_2, m''_3 = \pm l''_3$ so that $m''_1 + m''_2 + m''_3 = 0$. Since at least one parameter $p''_{i'} = 0$, the number of terms in the i'th double sum version of (4f) [related to (6a) and to the Kampé de Fériet [22, 23] function $F^{2:2}_{2:1}$] does not exceed

$$\widetilde{B}_{i'} = \min\left(\tfrac{1}{2}(p_{i'} + 1)_2, (k_{j'} + 1)(k_{k'} + 1)\right), \tag{40b}$$

although the ith version of (4f) or (6b) may be more preferable for small values of p_i for which $B_i < \widetilde{B}_{i'}$.

We may also express the isofactors of the CG coefficients for restriction $\mathrm{SO}(n) \supset \mathrm{SO}(n') \times \mathrm{SO}(n'')$ in terms of the isofactors of $3j$-symbols,

$$\begin{aligned} &\begin{bmatrix} l_1 & l_2 & l_3 \\ l'_1, l''_1 & l'_2, l''_2 & l'_3, l''_3 \end{bmatrix}_{(n:n'n'')} \\ &= (-1)^{\varphi} \left[\frac{d^{(n)}_{l_3}}{d^{(n')}_{l'_3} d^{(n'')}_{l''_3}}\right]^{1/2} \begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1, l''_1 & l'_2, l''_2 & l'_3, l''_3 \end{pmatrix}_{(n:n'n'')}, \end{aligned} \tag{41}$$

with the phase $\varphi = 0$ (since $l_3 - l'_3 - l''_3$ is even), when $\psi_n, \psi_{n'}, \psi_{n''}$ are taken to be equal to J, J', J'', respectively, in all the auxiliary $3j$-symbols (26a), in contrast to

$$\varphi = m''_3 \delta_{n''2} + m'_3 \delta_{n'2} + l''_3 \delta_{n''3} + l'_3 \delta_{n'3},$$

appearing when ψ_n is taken to be zero for $n \geq 4$. Again we need to replace, respectively, for $n'' = 2$ parameters l''_1, l''_2, l''_3 on the left-hand side by m''_1, m''_2, m''_3 so that $l''_1 = |m''_1|, l''_2 = |m''_2|, l''_3 = |m''_3|$ (with $m''_1 + m''_2 = m''_3$) and in the right-hand side by $m''_1, m''_2, -m''_3$, as well as for $n' = 2$ parameters l'_1, l'_2, l'_3 on the left-hand side by m'_1, m'_2, m'_3 so that $l'_1 = |m'_1|, l'_2 = |m'_2|, l'_3 = |m'_3|$ $(m'_1 + m'_2 = m'_3)$ and on the right-hand side by $m'_1, m'_2, -m'_3$.

Regarding the different triple-sum versions (9a)–(9d) of integrals involving triplets of Gegenbauer and Jacobi polynomials and comparing expressions (39) and (24), we derive the following duplication relation between the generic $\mathrm{SO}(n) \supset \mathrm{SO}(n-1)$ and special $\mathrm{SO}(2n+2) \supset \mathrm{SO}(n-1) \times \mathrm{SO}(n-1)$ isofactors of the $3j$-symbols:

$$\begin{aligned} \begin{pmatrix} 2l_1 & 2l_2 & 2l_3 \\ l'_1, l'_1 & l'_2, l'_2 & l'_3, l'_3 \end{pmatrix}_{(2n-2:n-1,n-1)} &= \prod_{a=1}^{3} \left[\frac{d^{(n)}_{l_a} d^{(n-1)}_{l'_a}}{d^{(2n-2)}_{2l_a}}\right]^{1/2} \\ &\times \begin{pmatrix} 2l_1 & 2l_2 & 2l_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1}_{2n-2} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}_{n} \\ &\times \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ 0 & 0 & 0 \end{pmatrix}_{n-1} \begin{pmatrix} l_1 & l_2 & l_3 \\ l'_1 & l'_2 & l'_3 \end{pmatrix}_{(n:n-1)}, \end{aligned} \tag{42}$$

with auxiliary $3j$-symbols (26a) of the canonical bases and the irrep dimensions appearing.

6 Basis states and coupling coefficients of the class-two representations of U(n)

Mixed tensor irreducible representations $[p+q, q^{n-2}, 0] \equiv [p, \dot{0}, -q]$ of $\mathrm{U}(n)$ containing scalar irrep $[q^{n-1}] \equiv [\dot{0}]$ of subgroup $\mathrm{U}(n-1)$ (with repeating zeros denoted by $\dot{0}$) are called class-two irreps [47, 48]; their canonical basis states for the chain $\mathrm{U}(n) \supset \mathrm{U}(n-1) \times \mathrm{U}(1) \supset \cdots \supset \mathrm{U}(2) \times \mathrm{U}(1) \supset \mathrm{U}(1)$ are labelled by the set

$$\begin{aligned} Q_{(n)} &= (p_{(n-1)}, q_{(n-1)}; Q'_{(n-1)}) \\ &= (p_{(n-1)}, q_{(n-1)}; p_{(n-2)}, q_{(n-2)}; \ldots, p_{(2)}, q_{(2)}; p_{(1)}), \end{aligned}$$

where

$$p = p_{(n)} \geq p_{(n-1)} \geq ... \geq p_{(2)} \geq 0 \quad \text{and} \quad q = q_{(n)} \geq q_{(n-1)} \geq ... \geq q_{(2)} \geq 0$$

are integers, with $p_{(2)} \geq p_{(1)} \geq -q_{(2)}$ in addition, and parameters

$$M_{(1)} = p_{(1)},\ M_{(2)} = p_{(2)} - q_{(2)} - p_{(1)}, ...,\ M_{(r)} = p_{(r)} - q_{(r)} - p_{(r-1)} + q_{(r-1)}$$

which correspond to irreps of subgroups U(1), beginning from the last one.

The dimension of representation space is

$$d^{(n)}_{[p,\dot{0},-q]} = \frac{(p+q+n-1)(p+1)_{n-2}(q+1)_{n-2}}{(n-1)!(n-2)!}. \tag{43}$$

Special matrix elements $D^{n[p,\dot{0},-q]}_{Q_{(n)};0}(g)$ of U(n) irrep $[p,\dot{0},-q]$ with zero as the second index for the scalar of subgroup U($n-1$) depend only on the rotation angles $\varphi_n, \varphi_{n-1}, ..., \varphi_2, \varphi_1$, where $0 \leq \varphi_i \leq 2\pi$ corresponds to the ith diagonal subgroup U(1) ($i = 1, 2, ..., n$), and $\theta_n, \theta_{n-1}, ..., \theta_3, \theta_2$, where $0 \leq \theta_r \leq \pi/2$, corresponds to the transformation

$$\begin{vmatrix} \cos\theta_r & \imath\sin\theta_r \\ \imath\sin\theta_r & \cos\theta_r \end{vmatrix}$$

in the plane of ($r-1$)st and rth coordinates ($r = 2, 3, ..., n$) and may be factorized as follows:

$$D^{n[p,\dot{0},-q]}_{Q_{(n)};0}(g) = \mathrm{e}^{\imath M_n \varphi_n} D^{n[p,\dot{0},-q]}_{[p',\dot{0},-q']0;0}(\theta_n) D^{n-1[p',\dot{0},-q']}_{Q_{(n-1)};0}(g') \tag{44}$$

with appropriate normalization in the case of integration over the group volume with measure

$$\frac{(n-1)!}{2\pi^n} \prod_{r=2}^{n} \sin^{2r-3}\theta_r \cos\theta_r \mathrm{d}\theta_r \prod_{i=1}^{n} \mathrm{d}\varphi_i.$$

Here $D^{n-1[p',\dot{0},-q']}_{Q_{(n-1)};0}(g')$ are the matrix elements of U($n-1$) irrep $[p',\dot{0},-q'] = [p_{(n-1)},\dot{0},-q_{(n-1)}]$ (with parameters obtained after omitting φ_n and θ_n). Special matrix elements of U(r) irreducible representation $[p,\dot{0},-q]$ with the U($r-1$) irrep labels $[p',\dot{0},-q']$ and 0 and SU($r-2$) irrep label 0 for rotation with angle θ_r in the (x_r, x_{r-1}) plane are written in terms of the D-matrices of SU(2) as follows:

$$\begin{aligned} D^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']0;0}(\theta_r) = {} & \left[(p+q+r-1)\, d^{(r-1)}_{[p',\dot{0},-q']}\right]^{1/2} \left[(r-1)\, d^{(r)}_{[p,\dot{0},-q]}\right]^{-1/2} \\ & \times (\imath\sin\theta_r)^{-r+2} P^{(p+q+r-2)/2}_{p'+(q-p+r-2)/2,-(p-q+r-2)/2-q'}(\cos 2\theta_r) \end{aligned} \tag{45}$$

and further, taking into account the identity $P^l_{m,n}(x) = P^l_{-n,-m}(x)$, in terms of the Jacobi polynomials

$$\begin{aligned} D^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']0;0}(\theta_r) = {} & \mathcal{N}^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']} \left[d^{(r-1)}_{[p',\dot{0},-q']} \left((r-1)\, d^{(r)}_{[p,\dot{0},-q]} \right)^{-1} \right]^{1/2} \\ & \times (\imath\sin\theta_r)^{p'+q'} (\cos\theta_r)^{|M|} P^{(L'+r-2,|M|)}_{K}(\cos 2\theta_r), \end{aligned} \tag{46}$$

where

$$K = \min(p-p', q-q'), \quad M = p-q-p'+q', \quad L' = p'+q'$$

and

$$\mathcal{N}^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']} = \left[\frac{(p+q+r-1)K!(p+q+r-2-K)!}{(|M|+K)!(p+q+r-2-|M|-K)!} \right]^{1/2}. \tag{47}$$

Factor $\imath^{p'+q'}$ (appearing also in [48], but absent in the generic expressions of D-matrix elements [43, 49]), ensures the complex conjugation relation

$$\overline{D^{r[p,\dot{0},-q]}_{[p',\dot{0},-q']0;0}(\theta_r)} = (-1)^{p'+q'} D^{r[q,\dot{0},-p]}_{[q',\dot{0},-p']0;0}(\theta_r), \tag{48}$$

in accordance with the SU(2) case and the system of phases of Baird and Biedenharn [50], which is correlated to the positive signs of the Gel'fand–Tsetlin matrix elements [42, 43, 51] of the U(n) generators $E_{r,r-1}$. Alternatively, the states $\Psi_{p'+q',p+q,M}(\theta_r)$, as defined in [4, 47] and related to the hyperspherical harmonics, correspond to the Jacobi polynomials with interchanged parameters α and β. Hence the variables are mutually reflected [here and in [4, 47] as $\cos 2\theta_r$ and $(-\cos 2\theta_r)$].

Using the integration over group (cf. [43, 47, 52]), the corresponding $3j$-symbols of the class-two irreps for the chain U(n)⊃U($n-1$)×U(1)⊃ ⋯ ⊃U(2)× U(1)⊃U(1) may be factorized as follows:

$$\begin{aligned}
&\sum\nolimits_\rho \begin{pmatrix} [p_1,\dot{0},-q_1] & [p_2,\dot{0},-q_2] & [p_3,\dot{0},-q_3] \\ Q_{1(n)} & Q_{2(n)} & Q_{3(n)} \end{pmatrix}_n^\rho \\
&\quad\times \begin{pmatrix} [p_1,\dot{0},-q_1] & [p_2,\dot{0},-q_2] & [p_3,\dot{0},-q_3] \\ [\dot{0}] & [\dot{0}] & [\dot{0}] \end{pmatrix}_n^\rho \\
&= \int\limits_{\mathrm{U}(n)} \mathrm{d}g D^{n[p_1,\dot{0},-q_1]}_{Q_{1(n)};0}(g) D^{n[p_2,\dot{0},-q_2]}_{Q_{2(n)};0}(g) D^{n[p_3,\dot{0},-q_3]}_{Q_{3(n)};0}(g) \qquad (49a)\\
&= \delta_{p_1+p_2+p_3,q_1+q_2+q_3}(-1)^{p'_1+p'_2+p'_3+K_1+K_2+K_3}(n-1)^{-1/2} \\
&\quad\times \prod\nolimits_{a=1}^3 \mathcal{N}^{n[p_a,\dot{0},-q_a]}_{[p'_a,\dot{0},-q'_a]} \left[d^{(n-1)}_{[p'_a,\dot{0},-q'_a]} \left(d^{(n)}_{[p_a,\dot{0},-q_a]} \right)^{-1} \right]^{1/2} \\
&\times \widetilde{\mathcal{I}} \begin{bmatrix} 0, n-2 & |M_1|, L'_1+n-2 & |M_2|, L'_2+n-2 & |M_3|, L'_3+n-2 \\ & K_1 & K_2 & K_3 \end{bmatrix} \\
&\quad\times \sum\nolimits_{\rho'} \begin{pmatrix} [p'_1,\dot{0},-q'_1] & [p'_2,\dot{0},-q'_2] & [p'_3,\dot{0},-q'_3] \\ Q'_{1(n-1)} & Q'_{2(n-1)} & Q'_{3(n-1)} \end{pmatrix}_{n-1}^{\rho'} \\
&\quad\times \begin{pmatrix} [p'_1,\dot{0},-q'_1] & [p'_2,\dot{0},-q'_2] & [p'_3,\dot{0},-q'_3] \\ [\dot{0}] & [\dot{0}] & [\dot{0}] \end{pmatrix}_{n-1}^{\rho'}. \qquad (49b)
\end{aligned}$$

Here ρ and ρ' are the multiplicity labels of the U(n) and U($n-1$) scalars in the decompositions $[p_1,\dot{0},-q_1] \otimes [p_2,\dot{0},-q_2] \otimes [p_3,\dot{0},-q_3]$ and $[p'_1,\dot{0},-q'_1] \otimes [p'_2,\dot{0},-q'_2] \otimes [p'_3,\dot{0},-q'_3]$. The integral involving the product of three Jacobi polynomials that appeared in (49b) also corresponds to the SO($2n$)⊃SO($2n-2$)×SO(2) isofactor of $3j$-symbol

$$\begin{pmatrix} p_1+q_1 & p_2+q_2 & p_3+q_3 \\ p'_1+q'_1, M_1 & p'_2+q'_2, M_2 & p'_3+q'_3, M_3 \end{pmatrix}_{(2n:2n-2,2)},$$

considered in previous section and may be expressed (after some permutation of parameters) as double sum by means of (6a) or (6b). For normalization of the corresponding $3j$-symbols of U(n)⊃U($n-1$) we may use square root of

$$\begin{aligned}
&\sum\nolimits_\rho \left[\begin{pmatrix} [p_1,\dot{0},-q_1] & [p_2,\dot{0},-q_2] & [p_3,\dot{0},-q_3] \\ [\dot{0}] & [\dot{0}] & [\dot{0}] \end{pmatrix}_n^\rho \right]^2 = \delta_{p_1+p_2+p_3,q_1+q_2+q_3} \\
&\quad\times (-1)^{\min(p_1,q_1)+\min(p_2,q_2)+\min(p_3,q_3)} \frac{(n-1)![(n-2)!]^2}{\prod_{a=1}^3 (\min(p_a,q_a)+1)_{n-2}} \\
&\quad\times \widetilde{\mathcal{I}} \begin{bmatrix} 0, n-2 & |p_1-q_1|, n-2 & |p_2-q_2|, n-2 & |p_3-q_3|, n-2 \\ & \min(p_1,q_1) & \min(p_2,q_2) & \min(p_3,q_3) \end{bmatrix}, \qquad (50)
\end{aligned}$$

with non-vanishing extreme $3j$-symbols in the left-hand side for a single value of the multiplicity label ρ, which is not correlated with the canonical [53, 54, 55] and other (see [56, 57, 58]) external labeling schemata of the coupling coefficients of U(n). In contrast to the particular $3j$-symbols (26a) of SO(n), equation (50) is summable only in the multiplicity-free cases. In addition to three double-sum versions of (6a) and (6b), the integral on the right-hand side of (50) may be also expressed as three different double-sum series by means of (4f), taking into account the symmetry relation (4b). Of course, (50) is always positive as an analogue of the denominator function of the SU(3) canonical tensor operators [53, 54, 55, 59].

Taking into account (48) we may also obtain expression for the Clebsch–Gordan coefficients of

class-two representation of U(n)

$$\begin{aligned}
&\textstyle\sum_\rho \left[\begin{array}{cc|c} [p_1,\dot{0},-q_1] & [p_2,\dot{0},-q_2] & [p,\dot{0},-q] \\ Q_{1(n)} & Q_{2(n)} & Q_{(n)} \end{array}\right]_n^\rho \\
&\quad\times \left[\begin{array}{cc|c} [p_1,\dot{0},-q_1] & [p_2,\dot{0},-q_2] & [p,\dot{0},-q] \\ [\dot{0}] & [\dot{0}] & [\dot{0}] \end{array}\right]_n^\rho \\
&= d^{(n)}_{[p,\dot{0},-q]} \int\limits_{\mathrm{U}(n)} \mathrm{d}g D^{n[p_1,\dot{0},-q_1]}_{Q_{1(n)};0}(g) D^{n[p_2,\dot{0},-q_2]}_{Q_{2(n)};0}(g) \overline{D^{n[p,\dot{0},-q]}_{Q_{(n)};0}(g)},
\end{aligned} \tag{51}$$

with the integrals involving the product of three Jacobi polynomials and the CG coefficients of U($n-1$) of the same type and some phase and irrep dimension factors. Particularly, we obtain the following expression for isofactors of special SU(3) Clebsch–Gordan coefficients (which perform the coupling of the SU(3)-hyperspherical harmonics):

$$\begin{aligned}
&\left[\begin{array}{ccc} (a'b') & (a''b'') & (ab)_0 \\ (z')i' & (z'')i'; & (z)i \end{array}\right] \\
&= \delta_{a'+a''-a,b'+b''-b}(-1)^{i'+i''-i+K_1+K_2-K}\,\tfrac{1}{2}\left[\frac{(2i'+1)(2i''+1)}{(2i+1)\,d^{(3)}_{(a'b')}d^{(3)}_{(a''b'')}}\right]^{1/2} \\
&\times\left\{\frac{(-1)^{\min(a',b')+\min(a'',b'')+\min(a,b)}}{(\min(a',b')+1)\,(\min(a'',b'')+1)\,(\min(a,b)+1)}\right. \\
&\times\left.\widetilde{\mathcal{I}}\left[\begin{array}{cccc} 0,1 & |a'-b'|,1 & |a''-b''|,1 & |a-b|,1 \\ & \min(a',b') & \min(a'',b'') & \min(a,b) \end{array}\right]\right\}^{-1/2} \\
&\times\mathcal{N}^{3[a',0,-b']}_{[i'-z',-i'-z']}\mathcal{N}^{3[a'',0,-b'']}_{[i''-z'',-i''-z'']}\mathcal{N}^{3[a,0,-b]}_{[i-z,-i-z]}\left[\begin{array}{ccc} i' & i'' & i \\ z' & z'' & z \end{array}\right] \\
&\times\widetilde{\mathcal{I}}\left[\begin{array}{cccc} 0,1 & |M'|,2i'+1 & |M''|,2i''+1 & |M|,2i+1 \\ & K' & K'' & K \end{array}\right].
\end{aligned} \tag{52}$$

Here $M=a-b+2z$, $K=\min(a+z-i,b-z-i)$ in the notation of [56, 58], with $(a\ b)$ for the mixed tensor irreps, where $a=p_{(3)}$, $b=q_{(3)}$ and the basis states are labelled by the isospin $i=\frac{1}{2}(p_{(2)}+q_{(2)})$, its projection $i_z=p_{(1)}-\frac{1}{2}(p_{(2)}-q_{(2)})$ and the parameter $z=\frac{1}{3}(b-a)-\frac{1}{2}y=\frac{1}{2}(q_{(2)}-p_{(2)})$ instead of the hypercharge $y=p_{(2)}-q_{(2)}-\frac{2}{3}(p_{(3)}-q_{(3)})$.

7 Weight shift operators of Sp(4) or SO(5)

Before considering the triple-sum series appeared [14] in the multiplicity-free isoscalar factors of Sp(4), we include some information about the basis states of symplectic group Sp(4) [SO(5)]. The irreducible representations of Sp(4) will be denoted by $\langle K\Lambda\rangle$, where the pairs of parameters $K=I_{\max},\Lambda=J_{\max}$ correspond to the maximal values of irreps I and J of the maximal subgroup SU(2)×SU(2) (see [14, 17, 40]) and to the irreps of SO(5) with the highest weight $[L_1L_2]=[K+\Lambda,K-\Lambda]$ and the branching rules $L_1\geq L_1'\geq L_2\geq|L_2'|$, where $L_1'=I+J$ and $L_2'=I-J$. The dimension of representation space of $\langle K\Lambda\rangle$ is

$$\tfrac{1}{6}(2K-2\Lambda+1)(2\Lambda+1)(2K+2)(2K+2\Lambda+3).$$

The infinitesimal operators (generators) of Sp(4) may be expressed as follows:

$$H_1=\tfrac{1}{2}(E_{11}-E_{22}),\quad F_{+0}=E_{12},\quad F_{-0}=E_{21}, \tag{53a}$$
$$H_2=\tfrac{1}{2}(E_{33}-E_{44}),\quad F_{0+}=E_{34},\quad F_{0-}=E_{43}, \tag{53b}$$
$$\begin{aligned} T_{++}&=-E_{14}-E_{32},\quad T_{--}=E_{41}+E_{23},\\ T_{+-}&=E_{13}-E_{42},\quad T_{-+}=E_{31}-E_{24}\end{aligned} \tag{53c}$$

in terms of generators of SU(4) which satisfy the defining relations $[E_{ik},E_{lm}]=\delta_{kl}E_{im}-\delta_{im}E_{lk}$. Operators (53a) and (53b) are generators of subgroups SU(2). Their matrix elements are well

known from the angular momentum theory [28, 29, 30]. Operators $T^{2\alpha}_{++}$, $T^{2\alpha}_{-+}$, $T^{2\alpha}_{+-}$ and $T^{2\alpha}_{--}$ form the extreme components of the double SU(2) irreducible tensor operator of rank α, α. The corresponding matrix elements may be expressed using the Wigner–Eckart theorem, e.g.

$$\left\langle \begin{matrix} \langle K\Lambda\rangle \\ I'M'J'N' \end{matrix} \middle| T^{2\alpha}_{-+} \middle| \begin{matrix} \langle K\Lambda\rangle \\ IMJN \end{matrix} \right\rangle = C^{I,\alpha,I'}_{M,-\alpha,M-\alpha} C^{J,\alpha,J'}_{N,\alpha,N+\alpha}$$
$$\times \frac{[(2I+1)(2J+1)]^{1/2}(J+J'-\alpha)!\nabla(\alpha II')\nabla(\alpha JJ')}{P(K\Lambda I'J')P(K\Lambda IJ)}$$
$$\times \sum_{i,j} \frac{(-1)^{J-I'-\alpha+i-j}(2i+1)P^2(K\Lambda ij)}{(2j+1)!(J+J'+\alpha-2j)!\nabla^2(j-J',I',i)\nabla^2(j-J,I,i)}, \tag{54}$$

where $\nabla(abc)$ is defined by (14a) and

$$P(K\Lambda IJ) = E(K+J,I,\Lambda)\nabla^{-1}(K-J,I,\Lambda),$$
$$E(abc) = [(a-b-c)!(a-b+c+1)!(a+b-c+1)!(a+b+c+2)!]^{1/2}. \tag{55}$$

The sum over i in asymmetric (with respect of the couples I, I' and J, J') expression (54) for the reduced matrix elements (cf. (7) of [14]) corresponds to a very well-poised ${}_9F_8(1)$ hypergeometric series [32, 33], but the attempting to rearrange it to a more suitable form was unsuccessful (e.g. in contrast with summations performed in [38]). Nevertheless, the reduced matrix elements (54) are summable in the SU(2) stretched cases (with $I' = I \pm \alpha$ or $J' = J \pm \alpha$), as well as for the symmetric irrep $\langle K0\rangle$ of Sp(4); (54) is proportional to the CG coefficient of SU(2) for $I+J = K+\Lambda$ or $I'+J' = K+\Lambda$, but ${}_3F_2(1)$ type series appearing for $|J-I| = K-\Lambda$ or $|J'+I'| = K-\Lambda$, as well as for the symmetric irrep $\langle\Lambda\Lambda\rangle$ in the general case, are not alternating.

The general weight lowering operators of Sp(4) introduced in [14] allow us to obtain the arbitrary basis states when acting into the highest weight state. Relation

$$\left| \begin{matrix} \langle K\Lambda\rangle \\ K-\alpha, K-\alpha, \Lambda+\alpha, \Lambda+\alpha \end{matrix} \right\rangle = \left[\frac{(2K-2\Lambda-2\alpha)!}{(2\alpha)!(2K-2\Lambda)!}\right]^{1/2} T^{2\alpha}_{-+} \left| \begin{matrix} \langle K\Lambda\rangle \\ KK\Lambda\Lambda \end{matrix} \right\rangle \tag{56}$$

(see (8) of [14]) may be used as the first step. More general basis state labelled by the chain Sp(4)⊃SU(2)×SU(2) may be obtained using expansion

$$\left| \begin{matrix} \langle K\Lambda\rangle \\ IIJJ \end{matrix} \right\rangle = \sum_{\alpha,\beta} Q[\langle K\Lambda\rangle IJ,\alpha\beta] F^{K-I-\alpha-\beta}_{-0} F^{\Lambda-J+\alpha-\beta}_{0-}$$
$$\times T^{2\beta}_{--} \left| \begin{matrix} \langle K\Lambda\rangle \\ K-\alpha, K-\alpha, \Lambda+\alpha, \Lambda+\alpha \end{matrix} \right\rangle \tag{57}$$

(see (9) of [14]). We express the expansion coefficient $Q[\langle K\Lambda\rangle IJ,\alpha\beta]$ in the following form:

$$Q[\langle K\Lambda\rangle IJ,\alpha\beta] = (-1)^{2\beta}\frac{E(K+\Lambda,I,J)\nabla(J,I,K-\Lambda)}{(2\beta)!(\Lambda+J+\alpha-\beta+1)!}$$
$$\times \left[\frac{(2I+1)!(2J+1)!(2\alpha)!}{(2K+1)!(2\Lambda)!(2K+2\Lambda+2)!(2K-2\Lambda-2\alpha)!}\right]^{1/2}$$
$$\times \sum_u \frac{(2K-2\Lambda-u)![(K-\Lambda+I+J-u+1)!]^{-1}}{u!(K-\Lambda-I+J-u)!(2\alpha-u)!(\Lambda-J-\alpha-\beta+u)!}, \tag{58}$$

appearing instead of the corresponding coefficient in (11) of [14] (or its alternative version), when the nonstandard CG coefficient of SU(2)

$$C^{I,J,K-\Lambda}_{\alpha+\beta-K-1,\Lambda+\alpha-\beta+1,\Lambda-K+2\alpha}$$

(cf. (10) of [14]) is expressed by means of (13.1c) of [28].

The weight shift relation (5.3) of [10]

$$\left|\begin{matrix}\langle K\Lambda\rangle\\ IIJJ\end{matrix}\right\rangle = \frac{\nabla(K-\Lambda,I,J)(K+\Lambda-I-J)!(K+\Lambda+I-J+1)!}{E(K+\Lambda,I,J)}$$
$$\times\left[\frac{(2\Lambda+1)!(2I+1)!(2J+1)!}{(2K+1)!(2K-2\Lambda)!}\right]^{1/2}\sum_j\left[\frac{(2K+2j+2)!}{(2j+1)(2\Lambda-2j)!}\right]^{1/2}$$
$$\times\frac{(-1)^{K-\Lambda-I-J+2j}}{(2J-2j)!(K-\Lambda-I-J+2j)!(K-\Lambda+I-J+2j+1)!}$$
$$\times F_{-0}^{K-\Lambda-I-J+2j}T_{-+}^{2J-2j}\left|\begin{matrix}\langle K\Lambda\rangle\\ K-\Lambda+j,K-\Lambda+j,j,j\end{matrix}\right\rangle \tag{59}$$

also will be useful in the next sections.

8 Semistretched isoscalar factors of the second kind of Sp(4) or SO(5)

Actually, the expression of [14, 18] for the semistretched isoscalar factors of the second kind[6] (with the coupled and resulting irrep parameters matching condition $K_1+K_2=K$) for the basis labelled by the chain Sp(4)⊃SU(2)×SU(2) has been derived in [14] using the weight lowering operator (57), expansion of the irreducible tensor operators, together with rearrangement formulas of the very well-poised hypergeometric ${}_6F_5(-1)$ series (cf. [32]), and may be presented in the following form:

$$\left[\begin{matrix}\langle K_1\Lambda_1\rangle & \langle K_2\Lambda_2\rangle & \langle K_1+K_2,\Lambda\rangle\\ I_1J_1 & I_2J_2 & I\,J\end{matrix}\right]$$
$$=(-1)^{\Lambda_1+\Lambda_2-\Lambda}\left[(2I_1+1)(2J_1+1)(2I_2+1)(2J_2+1)(2\Lambda+1)\right]^{1/2}$$
$$\times\left[\frac{\prod_{a=1}^2(2K_a-2\Lambda_a)!(2K_a+1)!(2K_a+2\Lambda_a+2)!}{(2K_1+2K_2-2\Lambda)!(2K_1+2K_2+1)!(2K_1+2K_2+2\Lambda+2)!}\right]^{1/2}$$
$$\times\frac{\nabla(K_1+K_2-\Lambda,I,J)\Delta(I_1I_2I)\Delta(J_1J_2J)\Delta(\Lambda_1\Lambda_2\Lambda)}{\prod_{a=1}^2E(K_a+\Lambda_a,I_a,J_a)\nabla(K_a-\Lambda_a,I_a,J_a)}$$
$$\times E(K_1+K_2+\Lambda,I,J)\,\widetilde{\mathcal{S}}\left[\begin{matrix}\alpha_0,\beta_0 & \alpha_1,\beta_1 & \alpha_2,\beta_2 & \alpha_3,\beta_3\\ & k_1 & k_2 & k_3\end{matrix}\right]. \tag{60}$$

In (60) and further we use the notations (14a), (55) and

$$\Delta(abc)=\left[\frac{(a+b-c)!(a-b+c)!(b+c-a)!}{(a+b+c+1)!}\right]^{1/2}, \tag{61}$$

and the triple sum $\widetilde{\mathcal{S}}[\cdots]$ in special parametrization:

$$\widetilde{\mathcal{S}}\left[\begin{matrix}\alpha_0,\beta_0 & \alpha_1,\beta_1 & \alpha_2,\beta_2 & \alpha_3,\beta_3\\ & k_1 & k_2 & k_3\end{matrix}\right]\equiv\widetilde{\mathcal{S}}\left[\begin{matrix}K_1 & j_1^1 & j_1^2 & j_1^3\\ K_2 & j_2^1 & j_2^2 & j_2^3\\ K_1+K_2 & j^1 & j^2 & j^3\end{matrix}\right]$$
$$=\sum_{z_1,z_2,z_3}\binom{p_0}{p_0'-z_1-z_2-z_3}\prod_{a=1}^3\frac{(-1)^{z_a}(-k_a-\alpha_a)_{z_a}(-k_a-\beta_a)_{k_a-z_a}}{z_a!(k_a-z_a)!} \tag{62a}$$
$$=\binom{-(2k_i+\alpha_i+\beta_i+2)}{-(k_i+\alpha_i+1)}\sum_{z_1,z_2,z_3}(-1)^{p_i-p_i''+z_1+z_2+z_3}\binom{p_i''}{p_i-z_1-z_2-z_3}$$
$$\times\frac{(k_i+1)_{z_i}(k_i+\beta_i+1)_{z_i}}{z_i!(2k_i+\alpha_i+\beta_i+2)_{z_i}}\prod_{a\neq i}\frac{(-k_a-\beta_a)_{z_a}(k_a+\alpha_a+\beta_a+1)_{k_a-z_a}}{z_a!(k_a-z_a)!} \tag{62b}$$

[6]Remind [14] that the semistretched isoscalar factors of the first kind (with the coupled and resulting irrep parameters matching condition $K_1-\Lambda_1+K_2-\Lambda_2=K-\Lambda$) are proportional to $9j$-coefficients of SU(2).

(with j_a^1, j_a^2, j_a^3, $a = 1, 2$, and j^1, j^2, j^3 corresponding to transposed I_a, J_a, Λ_a and I, J, Λ, respectively). The arguments of binomial coefficients are the non-negative integers. Here 11 parameters of the left-hand side of (60) or (62a) (corresponding to the array of the $11j$ coefficient [14, 18] of Sp(4)) are replaced by

$$\begin{gathered} k_1 = I_1 + I_2 - I, \quad k_2 = J_1 + J_2 - J, \quad k_3 = \Lambda_1 + \Lambda_2 - \Lambda; \\ \alpha_0 = -2K_2 - 1, \quad \alpha_1 = -2I_2 - 1, \quad \alpha_2 = -2J_2 - 1, \quad \alpha_3 = -2\Lambda_2 - 1; \\ \beta_0 = -2K_1 - 1, \quad \beta_1 = -2I_1 - 1, \quad \beta_2 = -2J_1 - 1, \quad \beta_3 = -2\Lambda_1 - 1. \end{gathered}$$

Although parameters α_j and β_j $(j = 0, 1, 2, 3)$ here are negative integers, arguments of binomial coefficients and 12 linear combinations

$$\begin{aligned} p_0' &= \tfrac{1}{2}(\beta_0 - \beta_1 - \beta_2 - \beta_3) - 1 = j_1^1 + j_1^2 + j_1^3 - K_1, \\ p_0'' &= \tfrac{1}{2}(\alpha_0 - \alpha_k - \alpha_i - \alpha_0) - 1 = j_2^1 + j_2^2 + j_2^3 - K_2, \\ p_0 &= p_0' + p_0'' - k_1 - k_2 - k_3 = j^1 + j^2 + j^3 - K_1 - K_2 \end{aligned} \tag{63a}$$

$$\begin{aligned} p_i' &= \tfrac{1}{2}(\beta_j + \beta_k - \beta_i - \beta_0) = K_1 - j_1^j - j_1^k + j_1^i, \\ p_i'' &= \tfrac{1}{2}(\alpha_j + \alpha_k - \alpha_i - \alpha_0) = K_2 - j_2^j - j_2^k + j_2^i, \\ p_i &= k_j + k_k - k_i + p_i' + p_i'' = K_1 + K_2 - j^j - j^k + j^i \end{aligned} \tag{63b}$$

$(i, j, k = 1, 2, 3)$ are non-negative integers, responding to the branching rules. Actually, expression (62b) may be written in three versions.

In spite of parameters α_a, β_a $(a = 0, 1, 2, 3)$ accepting the mutually excluding values in the sums $\widetilde{\mathcal{S}}[\cdots]$ and $\widetilde{\mathcal{I}}[\cdots]$, there is the one-to-one correspondence of the analytical continuation between series (62a) and (4c), as well as between series (62b) and (4f). In order to demonstrate it, the corresponding beta functions with parameters accepting all negative (integer or half-integer) values in (4c) and (4f) should be replaced by the binomial coefficients in (62a) and (62b), respectively. The possible zeros or poles may be disregarded, when the functions $\widetilde{\mathcal{S}}[\cdots]\binom{-\alpha_0-\beta_0-2}{-\alpha_0-1}^{-1}$ and $\widetilde{\mathcal{I}}[\cdots]\mathrm{B}^{-1}(\alpha_0+1, \beta_0+1)$ are considered, observing that the ratio of the binomial coefficients $\binom{-a-b-2}{-a-1}\binom{-c-d-2}{-c-1}^{-1}$ with negative integers a, b, c, d in equation (62a)–(62b) appeared from the ratio of the beta functions $\mathrm{B}(a+1, b+1)\mathrm{B}^{-1}(c+1, d+1)$ with parameters $a, b, c, d \geq -\frac{1}{2}$ in relation (4c)–(4f).

We see that the restrictions of summation parameters are more rich in (62a) and (62b) as in (4c)–(4f). For example, all three summation parameters are restricted by p_0', or by p_0'' in (62a), as well as by p_i, or by p_0' in (62b), taking into account that in this case $z_i \leq j_1^i - j_2^i + j^i$. Otherwise, the interval for the linear combination of summation parameters $z_1 + z_2 + z_3$ is restricted by p_0 in (62a), as well as by p_i'' in (62b). Hence, taking into account the symmetries there are five possibilities of the completely summable expressions for $\widetilde{\mathcal{S}}[\cdots]$ and seven cases when they turn into double sums, dissimilar with nine cases, related to the stretched $9j$ coefficients [21, 28].

As it was demonstrated in [17], the Sp(4) isofactors are invariant (up to a sign) or may be mutually related under elements of the substitution group, generated by the hook reflections and the hook permutations

$$\langle K\Lambda\rangle \to \langle -K-2, \Lambda\rangle, \tag{64a}$$

$$\langle K\Lambda\rangle \to \langle K, -\Lambda-1\rangle, \tag{64b}$$

$$\langle K\Lambda\rangle \to \langle \Lambda - 1/2, K + 1/2\rangle. \tag{64c}$$

Using the hook permutations $\langle K_1\Lambda_1\rangle \to \langle \Lambda_1 - 1/2, K_1 + 1/2\rangle$ and $\langle K\Lambda\rangle \to \langle \Lambda - 1/2, K + 1/2\rangle$ and "mirror" reflections $J_1 \to -J_1 - 1$ and $J \to -J - 1$ (cf. [28]) to (60), the expression for the non-standard semistretched isofactors of the second kind for the chain Sp(4)⊃SU(2)×SU(2) (with the coupled and resulting irreps matching condition $\Lambda_1 + K_2 = \Lambda$) may be presented in the following form:

$$\left[\begin{array}{ccc} \langle K_1\Lambda_1\rangle & \langle K_2\Lambda_2\rangle & \langle K, \Lambda_1 + K_2\rangle \\ I_1 J_1 & I_2 J_2 & I\,J \end{array}\right] = (-1)^{K_1+K_2-K-I_1-J_1+I+J}$$

$$\times\left[(2I_1+1)(2J_1+1)(2I_2+1)(2J_2+1)(2K+2)(2\Lambda_1)!(2K_1+2\Lambda_1+2)!\right]^{1/2}$$

$$\times\left[\frac{(2K_2-2\Lambda_2)!(2K_2+1)!(2K_2+2\Lambda_2+2)!(2K-2\Lambda_1-2K_2+1)!}{(2K_1-2\Lambda_1+1)!(2\Lambda_1+2K_2)!(2\Lambda_1+2K_2+2K+2)!}\right]^{1/2}$$
$$\times\frac{\nabla(K_1-\Lambda_1,I_1,J_1)\Delta(I_1I_2I)\Delta(J_1J_2J)\Delta(K_1+1/2,\Lambda_2,K+1/2)}{\nabla(K-\Lambda_1-K_2,I,J)\nabla(K_2-\Lambda_2,I_2,J_2)\prod_{a=1}^{2}E(K_a+\Lambda_a,I_a,J_a)}$$
$$\times E(\Lambda_1+K_2+K,I,J)\,\widetilde{\mathcal{S}}\left[\begin{array}{cccc}\alpha_0,\hat{\beta}_0 & \alpha_1,\beta_1 & \alpha_2,\hat{\beta}_2 & \alpha_3,\hat{\beta}_3\\ & k_1 & \hat{k}_2 & \hat{k}_3\end{array}\right] \tag{65}$$

(cf. (3.4) of [10]), where parameters of $\widetilde{\mathcal{S}}[\cdots]$ (expressed by means of (62b) with $i=1$) accept the values

$$k_1=I_1+I_2-I,\quad \hat{k}_2=J-J_1+J_2,\quad \hat{k}_3=K_1+\Lambda_2-K;$$
$$\alpha_0=-2K_2-1,\quad \alpha_1=-2I_2-1,\quad \alpha_2=-2J_2-1,\quad \alpha_3=-2\Lambda_2-1;$$
$$\hat{\beta}_0=-2\Lambda_1,\quad \beta_1=-2I_1-1,\quad \hat{\beta}_2=2J_1+1,\quad \hat{\beta}_3=-2K_1-2.$$

Taking into account that in this case

$$\hat{p}_1'=I_1+J_1+\Lambda_1-K_1,\quad p_1''=K_2-\Lambda_2+I_2-J_2,$$
$$\hat{p}_1=I+J+\Lambda_1+K_2-K$$

and summation parameter z_1 is restricted by $z_1\leq I_1-I_2+I$ (but condition $z_1\leq k_1$ cannot be regarded), all three summation parameters are restricted by conditions

$$\hat{p}_1-\min(K_a-\Lambda_a+I_a-J_a)\leq z_1+z_2+z_3\leq \hat{p}_1$$

(where $a=1$, or 2) and are fixed for $\hat{p}_1=0$, or $K_1-\Lambda_1+I_1-J_1=0$, when for $p_1''=0$ equation (65) turns into the double sum. Otherwise, when $\widetilde{\mathcal{S}}[\cdots]$ in (65) is expressed by means of (62a), all three summation parameters are restricted by the defining condition for the arguments of the binomial coefficients

$$\binom{K-\Lambda_1-K_2+I-J}{K_1-\Lambda_1+I_1-J_1-z_1-z_2-z_3}$$

and are fixed for $K_1-\Lambda_1+I_1-J_1=0$, or $I_2+J_2+\Lambda_2-K_2=0$. Hence, the expressions for (65) become simpler when responding to 8 branching rules of 12 possible.

Note that a restricting condition of the type $K_a-\Lambda_a+I_a-J_a$ ensures the double sum expressions for isofactors (60) or (65), e.g., in the case of a symmetric irrep $\langle\Lambda_a\Lambda_a\rangle$, whereas the second branching rule never causes the existence of any single sum expression. In general, new expressions for standard or non-standard triple series $\widetilde{\mathcal{S}}[\cdots]$ may be generated only by the elements of the Sp(4) and SU(2) substitution groups, which perform some permutation between parameters (63a)–(63b), i.e., the restricting conditions are not spoiled. For example, the expression for the non-standard semistretched isofactors of the second kind of Sp(4) with the coupled and resulting irreps matching condition $\Lambda_1-K_2=\Lambda$ (which correspond to isofactors (65) after interchange of $\langle K_1,\Lambda_1\rangle I_1,J_1$ and $\langle K,\Lambda_1+K_2\rangle I,J$) may be derived from (60) using the substitutions

$$\langle K_1\Lambda_1\rangle\to\langle-\Lambda_1-3/2,-K_1-3/2\rangle,\quad \langle K\Lambda\rangle\to\langle-\Lambda-3/2,-K-3/2\rangle, \tag{66a}$$

which do not spoil the restricting parameters (63a) in (62a) and (63b) in (62b), with the exception of p_3', p_3'' and p_3. Otherwise, parameters $\hat{p}_0'=K_1+\Lambda_1-I_1-J_1=p_3'$, p_0'' and $\hat{p}_0=K+\Lambda_1-K_2-I-J=p_3$ play the role of the restricting parameters of (62a), when the substitutions

$$\langle K_1\Lambda_1\rangle\to\langle-\Lambda_1-3/2,K_1+1/2\rangle,\quad \langle K\Lambda\rangle\to\langle-\Lambda-3/2,K+1/2\rangle,$$
$$I_1\to-I_1-1,\quad I\to-I-1,\quad J_1\to-J_1-1,\quad J\to-J-1, \tag{66b}$$

are used for (60).

As a consequence of rearrangement (16), an expression for special triple sum of the type (60) with coinciding the first two rows of the corresponding array is derived in the following form:

$$\widetilde{\mathbf{S}}\left[\begin{array}{cccc}K_1 & j_1^1 & j_1^2 & j_1^3\\ K_1 & j_1^1 & j_1^2 & j_1^3\\ 2K_1 & j^1 & j^2 & j^3\end{array}\right]$$

$$
\begin{aligned}
&= (-1)^{j_1^1+j_1^2+j_1^3-K_1-j^3}[1+(-1)^{j^1+j^2+j^3-2K_1}]2^{j^1+j^2-2K_1-1}(2j^3-1)! \\
&\times \frac{1}{j^3!}\prod_{a=1}^{2}\frac{(2j_1^a+j^a+1)!}{(2j^a+1)!(2j_1^a-j^a)!}\sum_{x_1,x_2,z_3}\binom{j_1^3-\frac{1}{2}(j^3+\delta_3)+x_3}{x_3} \\
&\times \frac{(-1)^{x_3}\left(-j_1^3-(j^3+\delta_3)/2\right)_{x_3}}{(-j^3+1/2)_{x_3}}\binom{\frac{1}{2}(\delta_1+\delta_2-\delta_3)}{K_1+j^3-\sum_{a=1}^{3}(\frac{1}{2}j^a-x_a)} \\
&\times \prod_{a=1}^{2}\frac{(j_1^a+(j^a+\delta_a)/2+1)_{x_a}}{(j^a+3/2)_{x_a}}\binom{j_1^a-\frac{1}{2}(j^a+\delta_a)}{x_a}. \qquad (67)
\end{aligned}
$$

Here $\delta_i = 0$ or 1, so that $j_1^i-(j^i+\delta_i)/2$ $(i=1,2,3)$ are integers.

Furthermore, an expression for the following more special triple-sum $\widetilde{\mathcal{S}}[\cdots]$ of the type (60) or (67) (with coinciding parameters $j_1^1=j_1^2$, $j_1^3=K_1$ in addition)

$$
\begin{aligned}
&\widetilde{\mathbf{S}}\begin{bmatrix} j_1^3 & j_1^1 & j_1^1 & j_1^3 \\ j_1^3 & j_1^1 & j_1^1 & j_1^3 \\ 2j_1^3 & j^1 & j^2 & j^3 \end{bmatrix} \\
&= \frac{(2j_1^3-2j_1^1)!\,\Gamma(1/2)\Gamma\left((j^1+j^2+j^3+1)/2-j_1^3\right)}{2^{4j_1^3+3}\prod_{a=1}^{3}j^a!\,\Gamma\left(j_1^3+(j^1+j^2+j^3+3)/2-j^a\right)} \\
&\times \sum_s \frac{(2j_1^1+j^1+1)!(2j_1^1+j^2+1)!(2j_1^3+j^3+1)!}{s!(2j_1^3-2j_1^1-s)!\,(j_1^3+(j^1-j^2-j^3)/2-s)!} \\
&\times \frac{[1+(-1)^{j^1+j^2+j^3-2j_1^3}](-1)^{2j_1^1+j_1^3-(j^1+j^2+j^3)/2}}{(j_1^3+(j^2-j^3-j^1)/2-s)!\,(2j_1^1-j_1^3+(j^3-j^1-j^2)/2+s)!} \\
&\times \frac{(2j_1^1+1/2)_s\Gamma(2j_1^3+3/2-s)}{(2j_1^1-j_1^3+(j^1+j^2+j^3)/2+s+1)!}, \qquad (68)
\end{aligned}
$$

was similarly derived as a consequence of rearrangement (17). Although this sum also corresponds to the balanced (Saalschützian) ${}_4F_3(1)$ type series [32, 33], it is not alternating (since it includes even numbers of gamma functions or factorials in numerator and denominator) and cannot be associated with the $6j$ coefficients of SU(2). Note, that the summable case of (68) with $j_1^3=j_1^1$ corresponds to (41) of [14].

9 Isofactors for coupling of two symmetric irreps of SO(n) in the canonical basis

Now, taking into account the complementary group relation, we may consider the most general isofactors of the CG coefficients of SO(n) ($n\ge 5$) for coupling of the two symmetric irreps in the canonical basis. In the phase system with $\psi_n=0$, we obtain the following relations (cf. [10, 15, 16]) for these isofactors:

$$
\begin{bmatrix} l_1 & l_2 & L_1, L_2 \\ l_1' & l_2' & L_1', L_2' \end{bmatrix}_{(n:n-1)} = \begin{bmatrix} l_1-r & l_2-r & L_1-r, L_2-r \\ l_1'-r & l_2'-r & L_1'-r, L_2'-r \end{bmatrix}_{(n+2r:n+2r-1)} \qquad (69a)
$$

$$
= \begin{bmatrix} \left\langle \frac{2l_1+n-5}{4}\,\frac{2l_1+n-5}{4} \right\rangle & \left\langle \frac{2l_2+n-5}{4}\,\frac{2l_1+n-5}{4} \right\rangle & \left\langle \frac{L_1+L_2+n-5}{2}\,\frac{L_1-L_2}{4} \right\rangle \\ \frac{2l_1'+n-5}{4},\frac{2l_1'+n-5}{4} & \frac{2l_2'+n-5}{4},\frac{2l_2'+n-5}{4} & \frac{L_1'+L_2'+n-5}{2},\frac{L_1'-L_2'}{2} \end{bmatrix}, \qquad (69b)
$$

which are also valid for half-integer values of r (as the analytical continuation relations).

Particularly, for $r=L_2'=L_2$ equation (24), together with (34), may be generalized as follows:

$$
\begin{aligned}
&\begin{bmatrix} l_1 & l_2 & L_1, L_2 \\ l_1' & l_2' & L_1', L_2 \end{bmatrix}_{(n:n-1)} = (-1)^{(l_1-l_1'-\delta_1+l_2-l_2'-\delta_2+L_1-L_1'-\delta)/2} \\
&\times\widetilde{\mathcal{I}}\begin{bmatrix} -\frac{1}{2}, L_2+\frac{n-3}{2} & \delta_1-\frac{1}{2}, l_1'+\frac{n-3}{2} & \delta_2-\frac{1}{2}, l_2'+\frac{n-3}{2} & \delta-\frac{1}{2}, L_1'+\frac{n-3}{2} \\ & \frac{1}{2}(l_1-l_1'-\delta_1) & \frac{1}{2}(l_2-l_2'-\delta_2) & \frac{1}{2}(L_1-L_1'-\delta) \end{bmatrix}
\end{aligned}
$$

$$\times\left[\frac{(2L_1+n-2)(L_1'-L_2)!(L_1+L_2+n-3)!\,\Gamma(L_2+n/2-1)}{8(L_1-L_2)!(L_1'+L_2+n-4)!\,\Gamma\left(L_2+(n-3)/2\right)}\right]^{1/2}$$
$$\times\frac{[\Gamma(1/2)(2l_1'+n-3)(2l_2'+n-3)]^{1/2}\,\widetilde{\nabla}_{n[0,1,2,3]}(l_1,l_2;L_1,L_2)}{\widetilde{\mathcal{H}}^{(n)}_{l_1:l_1',\delta_1}\widetilde{\mathcal{H}}^{(n)}_{l_2:l_2',\delta_2}\widetilde{\mathcal{H}}^{(n)}_{L_1:L_1',\delta}\widetilde{\nabla}_{n[0,1,2,3]}(l_1',l_2';L_1',L_2)} \tag{70}$$

(cf. (4.1) of [10]), where

$$\widetilde{\mathcal{H}}^{(n)}_{l_a:l_a',\delta_a}=\left[\frac{\Gamma\left(\frac{1}{2}(l_a-l_a'+\delta_a+1)\right)\Gamma\left(\frac{1}{2}(l_a+l_a'-\delta_a+n-1)\right)}{\left(\frac{1}{2}(l_a-l_a'-\delta_a)\right)!\,\Gamma\left(\frac{1}{2}(l_a+l_a'+\delta_a+n)-1\right)}\right]^{1/2}, \tag{71}$$

$\widetilde{\nabla}_{n[0,1,2,3]}(\cdots)$ is defined by (27) and integral $\widetilde{\mathcal{I}}[\cdots]$ may be expressed by means of (4f) (with i chosen 3, 2, or 1).

Further, for $L_1'=L_1$, the partial hook permutations $[L_1,L_2]\to[L_2-1,L_1+1]$ and $[L_1',L_2']\to[L_2'-1,L_1'+1]$ (cf. [60, 61]) allow us to transform equation (70) into

$$\left[\begin{array}{ccc} l_1 & l_2 & L_1,L_2 \\ l_1' & l_2' & L_1,L_2' \end{array}\right]_{(n:n-1)}=(-1)^{l_1-l_1'+(\delta_1+\delta_2-\delta)/2}$$
$$\times\widetilde{\mathcal{I}}\left[\begin{array}{cccc} -\frac{1}{2},L_1+\frac{n-1}{2} & \delta_1-\frac{1}{2},l_1'+\frac{n-3}{2} & \delta_2-\frac{1}{2},l_2'+\frac{n-3}{2} & \delta-\frac{1}{2},L_2'+\frac{n-5}{2} \\ & \frac{1}{2}(l_1-l_1'-\delta_1) & \frac{1}{2}(l_2-l_2'-\delta_2) & \frac{1}{2}(L_2-L_2'-\delta) \end{array}\right]$$
$$\times\left[\frac{(2L_2+n-4)(L_1-L_2+1)!(L_1+L_2+n-3)!\,\Gamma(L_1+n/2)}{8(L_1-L_2'+1)!(L_1+L_2'+n-4)!\,\Gamma\left(L_1+(n-1)/2\right)}\right]^{1/2}$$
$$\times\frac{[\Gamma(1/2)(2l_1'+n-3)(2l_2'+n-3)]^{1/2}\,\widetilde{\nabla}_{n-1[3,7]}(l_1',l_2';L_1,L_2')}{\widetilde{\mathcal{H}}^{(n)}_{l_1:l_1',\delta_1}\widetilde{\mathcal{H}}^{(n)}_{l_2:l_2',\delta_2}\widetilde{\mathcal{H}}^{(n-2)}_{L_2:L_2',\delta}\widetilde{\nabla}_{n[3,7]}(l_1,l_2;L_1,L_2)} \tag{72}$$

(cf. (4.4) of [10]), where

$$\widetilde{\nabla}_{n[3,7]}(a,b;e,f)=\left[\left(\tfrac{1}{2}(a-b+e-f)\right)!\left(\tfrac{1}{2}(b-a+e-f)\right)!\right.$$
$$\left.\times\,\Gamma\left(\tfrac{1}{2}(a-b+e+f+n)-1\right)\Gamma\left(\tfrac{1}{2}(b-a+e+f+n)-1\right)\right]^{1/2}$$
$$\times\left[\frac{\Gamma\left(\frac{1}{2}(a+b+e-f+n)\right)\left(\frac{1}{2}(a+b+e+f)+n-3\right)!}{\left(\frac{1}{2}(a+b-e-f)\right)!\,\Gamma\left(\frac{1}{2}(a+b-e+f+n)-1\right)}\right]^{1/2} \tag{73}$$

(cf. (28)), but the triple sum $\widetilde{\mathcal{I}}[\cdots]$ (with spoiled defining conditions $\alpha_a-\alpha_0\geq 0$ and $\beta_a-\beta_0\geq 0$ of integrals (4a)) may be expressed in this case only by means of (4f) with $i=3$ ($p_3=\frac{1}{2}(l_1+l_2-L_1-L_2)\geq 0$ and $p_3'=\frac{1}{2}(l_1'+l_2'-L_1-L_2')\geq 0$ being integers), as well as using less convenient expressions (4c)–(4e). The triple sum $\widetilde{\mathcal{I}}[\cdots]$ cannot be expressed as a version of (4f) with $i=1$, or 2, since p_1, p_1', p_2 and p_2' are negative in this case.

The rather complicated motivation of the phase choice in (72), passed over in [10], may be avoided, since equivalence of (72) for SO(5) with expression (33) of [14] may be proved after the transformation (by means of our relation (16)) of the triple sum

$$\widetilde{\mathcal{I}}\left[\begin{array}{cccc} L_1+2,L_1+2 & l_1'+1,l_1'+1 & l_2'+1,l_2'+1 & L_2',L_2' \\ & l_1-l_1' & l_2-l_2' & L_2-L_2' \end{array}\right]$$

that may be discerned in (33) of [14] using the parameters $L_1=K+\Lambda=I'+J'$, $L_2=K-\Lambda$, $l_1=2\Lambda_1$, $l_2=2\Lambda_2$, $l_1'=2I_1$, $l_2'=2I_2$ and $L_2'=2I'-K-\Lambda$. Note the phase factor $(-1)^{L_2-L_2'}$ that appears after interchange of the sets l_1,l_1' and l_2,l_2' in (72), in accordance with the number of performed antisymmetrizations.

In analogy with (23) of [14], we may derive the most general isofactors of Sp(4) in the SU(2)$\times$ SU(2) basis for coupling of the two symmetric irreps $\langle\Lambda_1\Lambda_1\rangle$ and $\langle\Lambda_2\Lambda_2\rangle$ from overlaps

$$\left\langle\begin{array}{c}\langle\Lambda_1\Lambda_1\rangle\\ I_1,I_1,I_1,J-I_2\end{array}\middle|\left\langle\begin{array}{c}\langle\Lambda_2\Lambda_2\rangle\\ I_2,I-I_1,I_2,I_2\end{array}\right|\middle|\begin{array}{c}\langle K\Lambda\rangle\\ II;JJ\end{array}\right\rangle, \tag{74}$$

expanded using the weight lowering operators of Sp(4) (57)–(58) or (59). Furthermore, using relation (69b) we express the most general isofactors of SO(n) for coupling of the two symmetric irreps in the canonical basis:

$$
\begin{aligned}
&\left[\begin{array}{ccc} l_1 & l_2 & L_1, L_2 \\ l_1' & l_2' & L_1', L_2' \end{array}\right]_{(n:n-1)} = \widetilde{\nabla}_{n-1[1,6]}(l_1', l_2'; L_1', L_2') \\
&\times \Bigg[\frac{(L_1+L_1'+n-3)!(L_1+L_2'+n-4)!(L_2+L_1'+n-4)!}{(L_1-L_2)!(L_1+L_2+n-4)!(2L_1+n-3)!} \\
&\times \frac{(L_1-L_2'+1)!(L_1-L_1')!(L_2-L_2')!(L_1'-L_2)!(l_1-l_1')!(l_2-l_2')!}{(L_2+L_2'+n-5)!(l_1+l_1'+n-3)!(l_2+l_2'+n-3)!}\Bigg]^{1/2} \\
&\times \sum_{l_1^0, l_2^0, L_2^0} \left[\frac{(2l_1'+n-3)(2l_2'+n-3)(l_1+l_1^0+n-3)!(l_2+l_2^0+n-3)!}{(2l_1^0+n-3)(2l_2^0+n-3)(l_1-l_1^0)!(l_2-l_2^0)!}\right]^{1/2} \\
&\times \frac{[(L_1+L_2^0+n-4)!(L_2-L_2^0)!(L_1-L_2^0+1)!]^{1/2}}{[(L_2+L_2^0+n-5)!]^{1/2}\widetilde{\nabla}_{n-1[1,6]}(l_1^0, l_2^0; L_1, L_2^0)(l_1^0-l_1')!(l_2^0-l_2')!} \\
&\times \sum_u \frac{(-1)^{(l_1'+l_2'-L_1'+L_2'+L_1-L_2^0-l_1^0-l_2^0)/2}(2L_2+n-5-u)!}{u!(L_2-L_2'-u)!(L_2-L_2^0-u)!(L_1'+L_2+n-4-u)!} \\
&\times \frac{\left[\left((l_1'+l_2'+L_1'-L_2'+L_1-L_2^0-l_1^0-l_2^0)/2+1\right)!\right]^{-1}}{((l_1'+l_2'-L_1'+L_2'+L_1+L_2^0-l_1^0-l_2^0)/2-L_2+u)!} \\
&\times \left[\begin{array}{ccc} l_1 & l_2 & L_1, L_2 \\ l_1^0 & l_2^0 & L_1, L_2^0 \end{array}\right]_{(n:n-1)},
\end{aligned} \tag{75}
$$

which are expanded in terms of the boundary (seed) isofactors (72).

In the second case we obtained other expression for the most general isofactors of SO(n) for coupling of the two symmetric irreps

$$
\begin{aligned}
&\left[\begin{array}{ccc} l_1 & l_2 & L_1, L_2 \\ l_1' & l_2' & L_1', L_2' \end{array}\right]_{(n:n-1)} = \widetilde{\nabla}_{n-1[1,3]}(l_1', l_2'; L_1', L_2') \\
&\times \Bigg[\frac{(L_1-L_2+1)!(L_1-L_1')!(L_2-L_2')!(L_1+L_2'+n-4)!}{(L_1+L_2+n-4)!(2L_2+n-5)!(L_1'-L_2)!(L_1-L_2'+1)!} \\
&\times \frac{(L_2+L_2'+n-5)!(L_2+L_1'+n-4)!(l_1-l_1')!(l_2-l_2')!}{(L_1+L_1'+n-3)!(l_1+l_1'+n-3)!(l_2+l_2'+n-3)!}\Bigg]^{1/2} \\
&\times \sum_{l_1^0, l_2^0, L_1^0} \left[\frac{(2l_1'+n-3)(2l_2'+n-3)(2l_2^0+n-3)(l_1+l_1^0+n-3)!}{(2l_1^0+n-3)(l_1-l_1^0)!(l_2+l_2^0+n-3)!(L_1-L_1^0)!}\right]^{1/2} \\
&\times \frac{[(l_2-l_2^0)!(L_1^0-L_2)!(L_1+L_1^0+n-3)!(L_2+L_1^0+n-4)!]^{1/2}}{2\widetilde{\nabla}_{n-1[1,3]}(l_1^0, l_2^0; L_1^0, L_2)(l_1^0-l_1')!(L_1^0-L_1')!(L_1^0+L_1'+n-4)!} \\
&\times \sum_v \frac{(-1)^{l_1^0-l_1'}(l_2+l_2'+n-3+u)!}{v!(l_2-l_2'-v)!(l_2'-l_2^0+v)!\,\Gamma\left(l_2'+(n-1)/2+v\right)} \\
&\times \frac{\Gamma\left((l_2'-l_1'-L_1'+L_2'+L_1^0-L_2+l_1^0+l_2^0+n-3)/2\right)}{((l_1'-l_2'+L_1'-L_2'-L_1^0+L_2-l_1^0+l_2^0)/2-v)!} \\
&\times \left[\begin{array}{ccc} l_1 & l_2 & L_1, L_2 \\ l_1^0 & l_2^0 & L_1^0, L_2 \end{array}\right]_{(n:n-1)},
\end{aligned} \tag{76}
$$

expanded in terms of the boundary (seed) isofactors (70). Note, that the restrictions for summation parameters l_1^0 and l_2^0 in (76) are different and an alternative version of it with mutually interchanged parameters l_1, l_1', l_1^0 and l_2, l_2', l_2^0 (but the same phase factor $(-1)^{l_1^0-l_1'}$) is possible. The total number of summation parameters in both expressions (75) and (76) is six, in contrast with seven in expansion (4.2) of [16] in terms of the boundary isofactors

$$
\left[\begin{array}{ccc} l_1 & l_2 & L_1, L_2 \\ l^0 & l^0 & L_2, L_2 \end{array}\right]_{(n:n-1)}.
$$

Isofactors of SO(n) for the semistretched coupling (with the coupled and resulting irrep parameters matching condition $l_1 + l_2 = L_1 + L_2$) may be expressed as the double sums, using relation (69b), together with (60) and $i = 2$ version of (62b). Otherwise, isofactors of SO(n) with the coupled and resulting irrep parameters matching condition $l_1 - l_2 = L_1 + L_2$ ($l_1 \geq l_2$) may be derived using relation (69b), together with (65) and the $i = 1$ version of (62b), after applying the symmetry relation (A.22) of [40] (interchange of $\langle K_1, \Lambda_1\rangle I_1, J_1$ and $\langle K, \Lambda_1 + K_2\rangle I, J$) to isofactors (65) of Sp(4).

10 Concluding remarks

In this paper, we reconsidered once more the $3j$-symbols and Clebsch–Gordan coefficients of the orthogonal SO(n) and unitary U(n) groups for all three representations corresponding to the (ultra)spherical or hyperspherical harmonics of these groups (i.e. irreps induced [42] by the scalar representations of the SO($n-1$) and U($n-1$) subgroups, respectively). For the corresponding isoscalar factors of the $3j$-symbols and coupling coefficients, the ordinary integrations involving triplets of the Gegenbauer and the Jacobi polynomials yield the more or less symmetric triple-sum expressions, however without the apparent triangle conditions. These conditions are visible and efficient only in here directly proved expressions (4f), (9c) and (13b), previously derived in [6, 15, 18] after complicated analytical continuation procedure of special Sp(4)$\supset$SU(2)$\times$SU(2) isofactors (cf. [14, 17]). Actually, only for a fixed integer shift parameter $p_i = \frac{1}{2}(l_j + l_k - l_i)$ it is evident that the corresponding integrals involving triplets of the Gegenbauer and the Jacobi polynomials are rational functions of remaining parameters. Practically, the concept of the canonical unit tensor operators (cf. section 21 of chapter 3 of [30]) for symmetric irreps of SO(n) may be formulated only under such a condition.

Similarly as special terminating double-hypergeometric series of Kampé de Fériet-type [21, 22, 23, 24, 62] correspond to the stretched $9j$ coefficients of SU(2), the definite terminating triple-hypergeometric series correspond either to the semistretched isofactors of the second kind [14] of Sp(4), or to the isofactors of the symmetric irreps of the orthogonal group SO(n) in the canonical and semicanonical (tree type) bases. Relation (2.6a)–(2.6c) of [18] (being significant within the framework of Sp(4) isofactors) is a triple-sum generalization of transformation formula (9) of [24] for terminating $F^{1:2,2}_{1:1,1}$ Kampé de Fériet series with a fixed single-integer non-positive parameter, restricting all summation parameters, although this termination condition is hidden in our equation (62a)–(62b). Our auxiliary expression (4e) for integrals corresponds to analytical continuation of the intermediate formula (2.6b) of [18] (or (26)–(27) of [14]) for special isofactors of Sp(4) or SO(5). However, relation (4c)–(4f) (important within the framework of SO(n) isofactors) cannot be associated with any transformation formula [24] for terminating $F^{1:2,2}_{1:1,1}$ Kampé de Fériet series with the same (single or double) parameters, restricting summation. Note the quite different procedures for generating the diversity of expressions for integrals and special isofactors of Sp(4) (variation of expressions for the Jacobi polynomials and use of the substitution group technique).

Expressions (62a) and (62b) corresponding to special Sp(4) isofactors are summable or turn into the terminating Kampé de Fériet [22, 23, 24] series $F^{2:2}_{2:1}$ for extreme basis states of Sp(4)$\supset$SU(2)$\times$ SU(2). Alternatively, in accordance with (11) and (4f), the expressions for special isofactors of SO(n) and SU(n) are summable in the case of the stretched couplings of the group representations and turn into the terminating Kampé de Fériet series $F^{2:2}_{2:1}$ for the irreps of subgroups in a stretched situation, including the generic cases for restrictions SO(n)$\supset$SO($n-1$), SO(n)$\supset$SO($n-2$)$\times$SO(2) and U(n)$\supset$U($n-1$). Taking into account the fact that the $F^{2:2}_{2:1}$ type series with five independent parameters also appeared as the denominator (normalization) functions of the SU(3) and $u_q(3)$ canonical tensor operators [53, 54, 55] (cf. (2.8) and section II of [59]), the q-extension of relation (6a)–(6b) from the classical SU(n) case may be suspected.

The expressions for special isofactors of SO(n) in terms of ${}_4F_3(1)$ series were helpful for rearrangement [38] of the fourfold [16, 35] and (corrected) [16] triple-sum expressions for the recoupling coefficients ($6j$-symbols) of symmetric irreps of SO(n) into the double $F^{1:4}_{1:3}$ type series, with the Regge type symmetry.

References

[1] N. J. Vilenkin, *Special Functions and the Theory of Group Representations* (American Mathematical Society, Providence, RI, 1968; Russian Original, Nauka, Moscow, 1965).

[2] A. M. Gavrilik, On the Clebsch–Gordan coefficients of the SO(n) and U(n) groups (*Preprint* ITP-73-104R, Kiev, 1973, in Russian).

[3] M. S. Kildyushov and G. I. Kuznetsov, Clebsch–Gordan coefficients for the class I representations of SO(n) group and for the class II representations of SU(n) group, (*Moscow Institute of Atomic Energy Preprint* no 2317, 1973, in Russian).

[4] G. I. Kuznetsov, S. S. Moskaliuk , Yu. F. Smirnov and V. P. Shelest, *Graphical Theory of Representations of Orthogonal Group and its Physical Applications* (Naukova Dumka, Kiev, 1992, in Russian).

[5] G. Junker, *J. Phys. A: Math. Gen.* **26**, 1649 (1993).

[6] E. Norvaišas and S. Ališauskas, *Liet. Fiz. Rinkinys* **14**, 443 (1974); [*Sov. Phys. -Coll. Lit. Fiz. Sb.* **14** (3), 1 (1974)].

[7] N. Ya. Vilenkin, *Mat. Sbornik* **68**, 432 (1965) [in Russian].

[8] S. Ališauskas and V. Vanagas, *Liet. Fiz. Rinkinys – Lit. Fiz. Sb.* **12**, 533 (1972) [in Russian].

[9] V. K. B. Kota, *J. Math. Phys.* **38**, 6639 (1997).

[10] E. Norvaišas and S. Ališauskas, *Liet. Fiz. Rinkinys* **14**, 715 (1974); [*Sov. Phys. -Coll. Lit. Fiz. Sb.* **14** (5), 5 (1974)].

[11] M. Moshinsky and C. Quesne, *J. Math. Phys.* **11**, 1631 (1970).

[12] M. Moshinsky and C. Quesne, *J. Math. Phys.* **12**, 1772 (1971).

[13] C. Quesne, *J. Math. Phys.* **14,** 388 (1973).

[14] S. Ališauskas and A. P. Jucys, *J. Math. Phys.* **12**, 594 (1971); [erratum, ibid. **13**, 575 (1972)].

[15] S. Ališauskas, *Fiz. Elem. Chast. At. Yad.* **14**, 1336 (1983); [*Sov. J. Part. Nucl.* **14**, 563 (1983)].

[16] S. Ališauskas, *J. Phys. A: Math. Gen.* **20**, 35 (1987).

[17] S. Ališauskas and A. P. Jucys, *J. Math. Phys.* **10**, 2227 (1969).

[18] S. Ališauskas, *J. Phys. A: Math. Gen.* **35**, 7323 (2002); Addendum, *ibid.* **37**, 1093 (2004); Preprint math-ph/0201048.

[19] A. A. Izmest'ev, G. S. Pogosyan, A. N. Sissakian and P. Winternitz, *J. Phys. A: Math. Gen.* **34**, 521 (2001).

[20] S. S. Moskaliuk and Yu. F. Smirnov, *Hadronic Journ.* **24**, 129 (2001).

[21] S. Ališauskas, *J. Math. Phys.* **41**, 7589 (2000).

[22] J. Kampé de Fériet, *C.R. Acad. Sci. Paris* **173**, 489 (1921).

[23] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques, Polynômes d'Hermite* (Gauthier-Villars, Paris, 1926).

[24] S. Lievens and J. Van der Jeugt, *J. Math. Phys.* **42**, 5417 (2001).

[25] G. Bateman and A. Erdelyi, *Higher Transcendental Functions*, Vol. 2 (McGraw-Hill, New York, 1953).

[26] M. Abramowitz and I. A. Stegun, (ed.), *Handbook of Mathematical Functions* (Dover, New York, 1965).

[27] G. E. Andreus, R. Askey and R. Roy, *Special Functions, Encyclopedia of Mathematics and Its Applications*, Vol. 71 (Cambridge University Press, Cambridge, 1999).

[28] A. P. Jucys and A. A. Bandzaitis, *Theory of Angular Momentum in Quantum Mechanics*, 2nd ed. (Mokslas, Vilnius, 1977) (in Russian).

[29] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, *Quantum Theory of Angular Momentum: Irreducible Tensors, Spherical Harmonics, Vector Coupling Coefficients,* $3nj$ *Symbols* (World Scientific, Singapore, 1988).

[30] L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics, Theory and Applications, Encyclopedia of Mathematics and its Applications*, Vol. 8 (Addison–Wesley, Reading, 1981).

[31] T. Regge, *Nuovo Cimento* **10**, 544 (1958).

[32] L. J. Slater, *Generalized Hypergeometric Series* (Cambridge University Press, Cambridge, 1966).

[33] G. Gasper and M. Rahman, *Basic Hypergeometric Series, Encyclopedia of Mathematics and Its Applications*, Vol. 35 (ed) Rota G-C (Cambridge University Press, Cambridge, 1990).

[34] K. Srinavasa Rao, J. Van der Jeugt, J. Raynal, R. Jagannathan and V. Rajeswari, *J. Phys. A: Math. Gen.* **25**, 861 (1992).

[35] M. Hormeß and G. Junker, *J. Phys. A: Math. Gen.* **32**, 4249 (1999).

[36] M. A. Lohe and L. C. Biedenharn, *SIAM J. Math. Anal.* **25**, 218 (1994).

[37] S.註išauskas, *J. Math. Phys.* **33**, 1983 (1992).

[38] S. Ališauskas, *J. Phys. A: Math. Gen.* **35**, 10229 (2002); Preprint math-ph/0206044.

[39] L. C. Biedenharn, *J. Math. Phys.* **2**, 433 (1961).

[40] K. T. Hecht, *Nucl. Phys.* **63**, 177 (1965).

[41] I. M. Gel'fand and M. L. Tsetlin, *Doklady Akad. Nauk SSSR* **71**, 1017 (1950) (in Russian).

[42] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications* (PWN–Polish Sci. Publ., Warszawa, 1977).

[43] A. U. Klimyk, *Matrix Elements and Clebsch–Gordan Coefficients of Group Representations* (Naukova Dumka, Kiev, 1979), (in Russian).

[44] A. M. Gavrilik and A. U. Klimyk, The representation matrix elements and Clebsch–Gordan coefficients of the rotation group (*Preprint* ITP-77-26E, Kiev, 1977).

[45] F. Pan and Y. F. Cao, *J. Math. Phys.* **29**, 2384 (1988).

[46] H. Z. Sun and D. Ruan, *J. Math. Phys.* **39**, 630 (1998).

[47] G. I. Kuznetsov, *Teor. Mat. Fiz.* **18**, 367 (1974).

[48] E. Norvaišas and S. Ališauskas, *Liet. Fiz. Rinkinys* **15**, 501 (1975); [*Sov. Phys. -Coll. Lit. Fiz. Sb.* **15** (4), 1 (1975)].

[49] N. Ya. Vilenkin, *Harmonic Analysis on Groups, Sb. Nauch. Trudov Mosk. Gos. Zaoch. Ped. Inst.*, Vol. 39, ed. N. Ya. Vilenkin (Moscow: MGZPI, Moscow, 1974), p. 77 (in Russian).

[50] B. Baird and L. C. Biedenharn, *J. Math. Phys.* **5**, 1723 (1964).

[51] I. M. Gel'fand and M. L. Tsetlin, *Doklady Akad. Nauk SSSR* **71**, 825 (1950) (in Russian).

[52] A. U. Klimyk and A. M. Gavrilik, *J. Math. Phys.* **20**, 1624 (1979).

[53] L. C. Biedenharn, J. D. Louck, E. Chacón and M. Ciftan, *J. Math. Phys.* **13**, 1957 (1972).

[54] L. C. Biedenharn, M. A. Lohe and J. D. Louck, *J. Math. Phys.* **26**, 1458 (1985).

[55] J. D. Louck, L. C. Biedenharn and M. A. Lohe, *J. Math. Phys.* **29**, 1106 (1988).

[56] S. Price Ališauskas, *J. Math. Phys.* **29**, 2551 (1988).

[57] S. Ališauskas, *J. Phys. A: Math. Gen.* **28**, 985 (1995).

[58] S. Ališauskas, *J. Phys. A: Math. Gen.* **29**, 2687 (1996).

[59] S. Ališauskas, *J. Math. Phys.* **40**, 5939 (1999).

[60] B. G. Wybourne, *Classical Groups for Physicists* (Willey, New York, 1974).

[61] S. Ališauskas and A. P. Jucys, *Rep. Math. Phys.* **5**, 7 (1974).

[62] J. Van der Jeugt, S. N. Pitre and K. Srinavasa Rao, *J. Phys. A: Math. Gen.* **27**, 5251 (1994).